5.4

10/17/18

Guess $y = |x|^r$

plug in

Solve $x^2y'' + \alpha xy' + \beta y = 0$. Let $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$ (case when $y = (-x)^r$ is similar).

$x^2x^{r-2}r(r-1) + \alpha x x^{r-1}r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$ for all x implies $r^2 + (\alpha - 1)r + \beta = 0$

Thus x^r is a solution iff $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Case 1: Two real roots, r_1, r_2.

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

Case 2: Two complex roots, $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\pm i\mu} = e^{ln(|x|^{\pm i\mu})} = e^{(\pm i\mu)ln|x|} = e^{i(\pm \mu ln|x|)}$

$$= cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)$$
$$= cos(\mu ln|x|) \pm isin(\mu ln|x|)$$

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2} = c_1|x|^{\lambda+i\mu} + c_2|x|^{\lambda-i\mu}$

$= |x|^\lambda(c_1|x|^{i\mu} + c_2|x|^{-i\mu})$

$= |x|^\lambda(c_1[cos(\mu ln|x|) + isin(\mu ln|x|)] + c_2[cos(\mu ln|x|) - isin(\mu ln|x|)])$

$= |x|^\lambda([c_1 + c_2]cos(\mu ln|x|) + i[c_1 - c_2]sin(\mu ln|x|))$

$= |x|^\lambda(k_1cos(\mu ln|x|) + k_2sin(\mu ln|x|))$

y $= k_1|x|^\lambda cos(\mu ln|x|) + k_2|x|^\lambda sin(\mu ln|x|)$

Case 3: one repeated root, $r_1 = \frac{-(\alpha-1)}{2}$. (i.e., $\sqrt{(\alpha-1)^2 - 4\beta} = 0$): $\blacksquare$

Thus $|x|^{r_1}$ is a solution. Find 2nd solution.

$y = c_1(|x|^{r_1}) + c_2(|x|^{r_1} ln|x|)$

$|x|^r ln|x|$

Method 1. Reduction of order: Suppose $y = u(x)|x|^{r_1}$ is a solution to $x^2y'' + \alpha xy' + \beta y = 0$. Plug in and determine $u(x)$

Method 2: Let $L(y) = x^2y'' + \alpha xy' + \beta y$ where $y' = \frac{dy}{dx}$.

$L(|x|^r) = |x|^r(r - r_1)^2$

$0 + 0$

$r_1 - r_1 \quad r_1 - r_1$

$\frac{\partial}{\partial r}[L(|x|^r)] = \frac{\partial}{\partial r}[|x|^r(r-r_1)^2] = (|x|^r)'(r-r_1)^2 + 2|x|^r(r-r_1) = 0$ if $r = r_1$.

Suppose x is constant with respect to r and all the partial derivatives are continuous. Then

$$\frac{\partial}{\partial r}[L(y)] = \frac{\partial}{\partial r}[x^2y'' + \alpha xy' + \beta y] = x^2\frac{\partial y''}{\partial r} + \alpha x\frac{\partial y'}{\partial r} + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial}{\partial r}[\frac{\partial^2 y}{\partial x^2}] + \alpha x\frac{\partial}{\partial r}[\frac{\partial y}{\partial x}] + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial^2}{\partial x^2}[\frac{\partial y}{\partial r}] + \alpha x\frac{\partial}{\partial x}[\frac{\partial y}{\partial r}] + \beta\frac{\partial y}{\partial r}$$
$$= L(\frac{\partial y}{\partial r}) \text{ for all } r$$

$\frac{\partial}{\partial r}[L(y)] = L(\frac{\partial y}{\partial r})$

$L(\frac{\partial |x|^r}{\partial r}) = \frac{\partial}{\partial r}[L(|x|^r)] = 0$ for $r = r_1$.

$\frac{\partial |x|^r}{\partial r} = \frac{\partial e^{ln|x|^r}}{\partial r} \frac{\partial e^{rln|x|}}{\partial r} = (e^{rln|x|})ln|x| = |x|^r ln|x|$

Thus $|x|^{r_1}ln|x|$ is a solution.

Thus general solution is $y = c_1|x|^{r_1} + c_2|x|^{r_1}ln|x|$

since by the Wronskian, $|x|^{r_1}$ and $|x|^{r_1}ln|x|$ are linearly independent. Suppose $x > 0$ and $r_1 \neq 0$.

$$\begin{vmatrix} x^{r_1} & x^{r_1}ln|x| \\ r_1x^{r_1-1} & r_1x^{r_1-1}ln|x| + x^{r_1-1} \end{vmatrix}$$

$$= x^{r_1}(r_1x^{r_1-1}ln|x| + x^{r_1-1}) - x^{r_1}ln|x|r_1x^{r_1-1}$$

$$= x^{2r_1-1}[r_1ln|x| + 1 - ln|x|r_1] = x^{2r_1-1} \neq 0 \text{ for } x \neq 0$$

Other cases for Wronskian are similar.