5.1 Review of Power Series.

Definition: $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = lim_{n\to\infty} \Sigma_{n=0}^{k} a_n(x-x_0)^n$

Taylor's Theorem

Suppose f has $n+1$ continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + R_{n+1}(x)$$

where the error term $R_{n+1}(x)$ satisfies $R_{n+1}(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

The *infinite* Taylor series converges to f,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ if and only if } \lim_{n\to\infty} R_n(x) = 0.$$

Defn: The function f is said to be **analytic** at a is its Taylor series expansion about $x = a$ has a positive radius of convergence.

1.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges at the point x if and only if $lim_{n\to\infty}\Sigma_{n=0}^{k} a_n(x-x_0)^n$ exists at the point x.

2.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely at the point x if and only if $\Sigma_{n=0}^{\infty}|a_n||x-x_0|^n$ converges at the point x

If a series converges absolutely, then it also converges.

3.) Ratio test for absolute convergence:

Let $L = lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$

$lim_{n\to\infty}|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}| = |x-x_0| lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = |x-x_0|L$

approx works in interval $(x_0-\frac{1}{L}, x_0+\frac{1}{L})$

The power series converges at the value x if $|x-x_0| < \frac{1}{L}$

$x_0-\frac{1}{L}$ $\quad x_0 \quad$ $x_0+\frac{1}{L}$

The power series diverges at the value x if $|x-x_0| > \frac{1}{L}$

The ratio test give no info at the value x if $|x-x_0| = \frac{1}{L}$

Note $\frac{1}{L}$ is the **radius of convergence**.

handle as special case

div $\qquad$ div

Solve $y'' - 4y' + 4y = 0$

Using quick 3.4 method. Guess $y = e^{rt}$ and plug into equation to find $r^2 - 4r + 4 = 0$. Thus $(r-2)^2 = 0$. Hence $r = 2$. Therefore general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$.

Use LONG 5.2 method (normally use this method only when other shorter methods don't exist) to find solution for values near $x_0 = 0$.

Suppose the solution $y = f(x)$ is analytic at $x_0 = 0$.

That is $f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$ for x near $x_0 = 0$. Taylor series

Thus there are constants $a_n = \frac{f^{(n)}(0)}{n!}$ such that,

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x-0)^n = \Sigma_{n=0}^{\infty} a_n x^n.$$

plug to find a_n

Find a recursive formula for the constants of the series solution to $y'' - 4y' + 4y = 0$ near $x_0 = 0$

We will determine these constants a_n by plugging f into the ODE.

$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$, $f'(x) = \Sigma_{n=1}^{\infty} a_n n x^{n-1}$, $f''(x) = \Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

$\Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 4\Sigma_{n=1}^{\infty} a_n n x^{n-1} + 4\Sigma_{n=0}^{\infty} a_n x^n = 0.$ ← solve for a_n by combining like terms

$\Sigma_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 4\Sigma_{n=0}^{\infty} a_{n+1}(n+1)x^n + 4\Sigma_{n=0}^{\infty} a_n x^n = 0.$

$\Sigma_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n] x^n = 0. = \sum_{n=0}^{\infty} 0 x^n$

$a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n = 0.$

$a_{n+2} = \frac{4a_{n+1}(n+1) - 4a_n}{(n+2)(n+1)}$. ← solve for a_{n+2} ← largest subscript

Hence the recursive formula (if know previous terms, can determine later terms) is

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$$

$\sum_{n=0}^{\infty} a_n x^n$ $a_0 = a_0$ $a_1 = a_1$

Given the recursive formula, $a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$, determine a_n.

Determine formula for a_k by noticing patterns. Note: It is easier to notice patterns if you do NOT simplify too much.

Find the first 6 terms of the series solution

$n = 0:\quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right)$

$n = 1:\quad a_3 = 4\left(\frac{2a_2 - a_1}{(3)(2)}\right) = 4\left(\frac{(2)(4)\left(\frac{a_1 - a_0}{(2)(1)}\right) - a_1}{(3)(2)}\right) = 4\left(\frac{4(a_1 - a_0) - a_1}{(3)(2)}\right) = 4\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2:\quad a_4 = 4\left(\frac{3a_3 - a_2}{(4)(3)}\right) = 4\left(\frac{3(4)\left(\frac{3a_1 - 4a_0}{3!}\right) - 4\left(\frac{a_1 - a_0}{2!}\right)}{(4)(3)}\right) = 4\left(\frac{3\left(\frac{3a_1 - 4a_0}{3!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right)$

$= 4\left(\frac{\left(\frac{3a_1 - 4a_0}{2!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right) = 4\left(\frac{(3a_1 - 4a_0) - (a_1 - a_0)}{3!}\right) = 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)$

$n = 3:\quad a_5 = 4\left(\frac{(4)a_4 - a_3}{(5)(4)}\right) = 4\left(\frac{(4)4\left(\frac{2a_1 - 3a_0}{3!}\right) - 4\left(\frac{3a_1 - 4a_0}{3!}\right)}{(5)(4)}\right)$

$= 4\left(\frac{4\left(\frac{2a_1 - 3a_0}{3!}\right) - \left(\frac{3a_1 - 4a_0}{3!}\right)}{5}\right) = 4\left(\frac{4(2a_1 - 3a_0) - (3a_1 - 4a_0)}{5(3!)}\right) = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)$

$f(x) \sim a_0 + a_1 x + 4\left(\frac{a_1 - a_0}{2!}\right)x^2 + 4\left(\frac{3a_1 - 4a_0}{3!}\right)x^3 + 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)x^4 + 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)x^5$

Recall $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$ for linearly independent solutions ϕ_0 and ϕ_1 to equation $y'' - 4y' + 4y = 0$. $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Find the first 5 terms in each of the 2 solns $y = \phi_0(x)$ and $y = \phi_1(x)$

$\phi_0 \sim 1 + 4\left(\frac{-1}{2!}\right)x^2 + 4\left(\frac{-4}{3!}\right)x^3 + 4\left(\frac{-3}{(3!)}\right)x^4 + 4\left(\frac{-8}{5(3!)}\right)x^5$

$\phi_1 \sim x + 4\left(\frac{1}{2!}\right)x^2 + 4\left(\frac{3}{3!}\right)x^3 + 4\left(\frac{2}{(3!)}\right)x^4 + 4\left(\frac{5}{5(3!)}\right)x^5$

$n = 0:\quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right) = 2\left(\frac{2a_1 - 2a_0}{2!}\right)$

$n = 1:\quad a_3 = = 4\left(\frac{3a_1 - 4a_0}{3!}\right) = 2^2\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2:\quad a_4 = 4\left(\frac{2a_1 - 3a_0}{3!}\right) = 16\left(\frac{2a_1 - 3a_0}{4!}\right) = 8\left(\frac{4a_1 - 6a_0}{4!}\right) = 2^3\left(\frac{4a_1 - 6a_0}{4!}\right)$

$n = 3:\quad a_5 = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right) = 16\left(\frac{5a_1 - 8a_0}{5!}\right) = 2^4\left(\frac{5a_1 - 8a_0}{5!}\right)$

Hence it appears $a_k = \frac{2^{k-1}(ka_1 - 2(k-1)a_0)}{k!}$