1) Solve homog $y'' - 4y' - 5y = 0$

$r^2 - 4r - 5 = 0$ $\quad y = c_1e^{5t} + c_2e^{-t}$

2.) Find one solution to non-homogeneous eq'n:

Find a solution to $ay'' + by' + cy = 4sin(3t)$:

$\boxed{\text{Guess } y = Asin(3t) + Bcos(3t)}$

$$y' = 3Acos(3t) - 3Bsin(3t)$$

$$y'' = -9Asin(3t) - 9Bcos(3t)$$

$y'' - 4y' - 5y = 4sin(3t)$

plug in to find A & B

$$\begin{array}{rcl} -9Asin(3t) & - & 9Bcos(3t) \\ 12Bsin(3t) & - & 12Acos(3t) \\ -5Asin(3t) & - & 5cos(3t) \\ \hline (12B - 14A)sin(3t) & - & (-14B - 12A)cos(3t) = 4sin(3t) \end{array}$$

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$12B - 14A = 4$ and $-14B - 12A = 0$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the nonhomogeneous equation.

Thus the general solution to the 2nd order linear non-homogeneous equation is

$$\underbrace{\underbrace{y = c_1e^{-t} + c_2e^{5t}}_{\text{homog}} + \underbrace{-(\tfrac{14}{85})sin(3t) + \tfrac{12}{85}cos(3t)}_{\text{1 nonhomog}}}_{\text{general nonhomog soln}}$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to separately satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t)$, $\underline{y(0) = 6, y'(0) = 7}$.

General solution: $y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$

Thus $y' = -c_1e^{-t} + 5c_2e^{5t} - (\frac{42}{85})cos(3t) - \frac{36}{85}sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \qquad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \qquad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$.

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6$.

$$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7.$$

$(e^{-t})'' - 4(e^{-t})' - 5(e^{-t}) = 0$ and $(e^{5t})'' - 4(e^{5t})' - 5(e^{5t}) = 0$

$f \& g$ Linearly indep if $c_1 f + c_2 g = 0$ has unique soln $c_1 = 0 = c_2$

3.3: Linear Independence and the Wronskian

Defn: f and g are linearly dependent if there exists constants c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$ and

$$c_1 f(t) + c_2 g(t) = 0 \text{ for all } t \in (a,b)$$

Thm 3.3.1: If $f : (a,b) \to R$ and $g(a,b) \to R$ are differentiable functions on (a, b) and if $W(f,g)(t_0) \neq 0$ for some $t_0 \in (a,b)$, then f and g are linearly independent on (a,b). Moreover, if f and g are linearly dependent on (a,b), then $W(f,g)(t) = 0$ for all $t \in (a,b)$

If $c_1 f(t) + c_2 g(t) = 0$ for all t, then $c_1 f'(t) + c_2 g'(t) = 0$

Solve the following linear system of equations for c_1, c_2

$$c_1 f(t_0) + c_2 g(t_0) = 0$$
$$c_1 f'(t_0) + c_2 g'(t_0) = 0$$

$$\underbrace{\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix}}_{\text{coef matrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when soln for $c_1 \& c_2$ in an IVP

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t) \; [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\boxed{\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)}$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

$$L(h) = 0$$

Since ψ is a solution to $ay'' + by' + cy = g(t)$, [*]

$$L(\psi) = g$$

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

$$L(\psi + h) = L(\psi) + L(h)$$
$$= g + 0 = g$$

Since γ a solution to $ay'' + by' + cy = g(t)$,

$$L(\gamma) = g$$

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

$$L(\gamma - \psi) = L(\gamma) - L(\psi)$$
$$= g - g = 0$$

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = c_1\phi_1 + c_2\phi_2$$

Thus $\boxed{\underbrace{\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t).}_{\text{general non homog soln}}}$

All non homog solns look like this

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $\boxed{ay'' + by' + cy = g_1(t) + g_2(t)}$

Proof: Let $L(f) = af'' + bf' + cf$.

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

$$L(f_1) = g_1$$

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

$$L(f_2) = g_2$$

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

$$L(f_1 + f_2) = L(f_1) + L(f_2)$$
$$= g_1 + g_2$$

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Solve homog first

Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $y'' - 4y' - 5y = 0$ is $y = c_1 e^{-t} + c_2 e^{5t}$

since $r^2 - 4r - 5 = (r-5)(r+1) = 0$

1.) $y'' - 4y' - 5y = 4e^{2t}$

Guess: $y = Ae^{2t}$

$4Ae^{2t} - 4(2Ae^{2t}) - 5(Ae^{2t}) = 4e^{2t}$

2a.) $y'' - 4y' - 5y = t^2 - 2t + 1$

Guess: $y = At^2 + Bt + C$

$2A - 4(2At + B) - 5(At^2 + Bt + C)$

$2A - 4B - 5C = 1$

$-8At - 5Bt = -2t$

$-5At^2 = t^2$

2a.) $y'' - 4y' - 5y = t^2$

Guess: $y = At^2 + Bt + C$

2c.) $y'' - 4y' - 5y =$ a degree 2 polynomial

Guess: $y = At^2 + Bt + C$

3a.) $y'' - 4y' - 5y = 30$

Guess: $y = C$ $\quad C = -6$

4a.) $y'' - 4y' - 5y = 4sin(3t)$

Guess: $y = A\sin(3t) + B\cos(3t)$

4b.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

Guess: $y = A\sin(3t) + B\cos(3t)$

4c.) $y'' - 4y' - 5y = 5cos(3t)$

Guess: $y = A\cos(3t) + B\sin(3t)$

5.) $y'' - 4y' - 5y = 4e^{-t}$

Guess: $\cancel{y = Ae^{-t}}$ $\quad$ $y \neq Ae^{-t}$ for non hom since it is homo soln $\Longrightarrow$ so instead guess $y = Ate^{-t}$

0 homog

or $y = (At + \cancel{B})e^{-t}$

homo

$y = c_1\phi_1 + c_2\phi_2 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$) solve 4 seperate non homog eqns & then combine them

6.) $y'' - 4y' - 5y = e^t + e^{-t} + 2t^3 + 3t^2 + 4sin(3t) + 5cos(3t)$

Guess: $y = A_1 e^t + A_2 t e^{-t} + A_3 t^3 + Bt^2 + Ct + D$
$+ A_4 \sin 3t + B_4 \cos(3t)$

7.) $y'' - 4y' - 5y = e^t + e^{-t} + 2t^3 + 3t^2 + 4sin(3t) + 5cos(t)$

Guess: $y = A_1 e^t + Ate^{-t} + A_3 t^3 + B_3 t^2 + C_3 t + D_3$
$+ A_4 \sin 3t + B_4 \cos 3t + A_5 \sin(t) + B_5 \cos(t)$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

Guess: $y = (At^2 + Bt + C)e^{2t}$

AD BD CD constant

Note homogeneous solution to $\boxed{y'' - 6y' + 9y = 0 \quad \text{is} \quad y = c_1 e^{3t} + c_2 t e^{3t}}$
since $r^2 - 6r + 9 = (r-3)(r-3) = 0$

Solve homog first

9.) $y'' - 6y' + 9y = 7e^{3t}$

Guess: $y = At^2 e^{3t}$

10.) $y'' - 6y' + 9y = 7e^{-3t}$

Guess: $y = Ae^{-3t}$

Some special cases: $y = A\sin(3t) + \cancel{B\cos(3t)}$ since not homog

no y' term

11.) $y'' - 5y = 4sin(3t)$

Best Guess: $y = A\sin(3t)$

since no y term

don't need

12.) $y'' - 4y' = t^2 - 2t + 1$

Guess: $y = At^3 + Bt^2 + Ct + D$

13) $y'' - 4y' - 5y = t\sin t \Rightarrow y = (At+B)(C\sin t + D\cos t)$

Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $y'' - 4y' - 5y = 0$ is $y = c_1e^{-t} + c_2e^{5t}$

$$\text{since } r^2 - 4r - 5 = (r-5)(r+1) = 0$$

1.) $y'' - 4y' - 5y = 4e^{2t}$

Guess: $\underline{\quad y \ = \ Ae^{2t} \quad}$

We want to plug in a solution for y into the left-hand side (LHS) of the equation that will give us the right-hand side (RHS) of the equation. In this case we need the output of the LHS to be a multiple of e^{2t} (in particular the output of the LHS when pluggin in y needs to be the RHS which is $4e^{2t}$).

Thus we guess $y = Ae^{2t}$. Plugging this into the LHS, we can solve for A so that we get the RHS, $4e^{2t}$, thus finding a non-homogeneous solution.

2a.) $y'' - 4y' - 5y = t^2 - 2t + 1$

Guess: $\underline{\quad y \ = \ At^2 + Bt + C \quad}$

2b.) $y'' - 4y' - 5y = t^2$

Guess: $\underline{\quad y \ = \ At^2 + Bt + C \quad}$

2c.) $y'' - 4y' - 5y =$ a degree 2 polynomial

Guess: $\underline{\quad y \ = \ At^2 + Bt + C \quad}$

Note that the non-homog solution guess is the same for 2a, 2b, 2c. In each case, we need to guess a solution such that when we plug it into the LHS we get the RHS, a degree 2 polynomial. Thus our guess is a degree 2 polynomial (but compare this to example 12). Note we need $y = At^2 + Bt + C$ even in case 2b. Make sure you understand why $y = At^2$ won't work.

3a.) $y'' - 4y' - 5y = 30$

Guess: $\underline{\quad y \ = \ A \quad}$

We have a constant on the RHS, so I guess a constant (but again compare to example 12). If you are observant, you may note that a non-homog solution is $y = -6$)

4a.) $y'' - 4y' - 5y = 4sin(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

4b.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

4c.) $y'' - 4y' - 5y = 5cos(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

Note that the non-homog solution guess is the same for 4a, 4b, 4c. If we plug in $y = Asin(3t)$, the output will contain both $sin(3t)$ and $cos(3t)$ terms. Thus I need to include both these terms in my guess. Compare to example 11.

5.) $y'' - 4y' - 5y = 4e^{-t}$

Guess: $\underline{\quad y \;=\; Ate^{-t} \quad}$

Note $y = Ae^{-t}$ is a homogeneous solution. Thus if I plug in it, I will get 0. But I want the RHS, $4e^{-t}$. When a guess doesn't work because it is a homogeneous solution, multiple by t.

Sidenote: this trick works because when you plug it in, you must use the product rule; the homogeneous part e^{-t} of $y = Ate^{-t}$ will result in a number of cancellations, but the t part will give you terms that don't cancel out and whose sum is the RHS.

Observe $y = Ate^{-t}$, $\quad y' = Ae^{-t} - Ate^{-t}$, $\quad y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

Thus $y'' - 4y' - 5y = -2Ae^{-t} + Ate^{-t} - 4(Ae^{-t} - Ate^{-t}) - 5(Ate^{-t})$

$= -2Ae^{-t} - 4Ae^{-t} + At(e^{-t} + 4e^{-t} - 5e^{-t}) = -6Ae^{-t} + At(0) = 4e^{-t}$ when $A = -\frac{2}{3}$

DO NOT FORGET THE PRODUCT RULE!!!!

6.) $y'' - 4y' - 5y = (e^t) + (e^{-t}) + (2t^3 + 3t^2) + (4sin(3t) + 5cos(3t))$

Guess: $\underline{\;y \;=\; (A_1e^t) + (A_2te^{-t}) + (A_3t^3 + B_3t^2 + C_3t + D_3) + (A_4sin(3t) + B_4cos(3t))\;}$

Note if we wanted to find a non-homogeneous solution, we would need to determine all our undetermined coefficients. Note we have 8 undetermined coefficients. Instead of solving for them all at once (which would require 8 equations for the 8 unknowns), it is easier to divide finding a non-homogeneous solution into 4 simpler parts indicated by the parenthesis and subscripts as described below:

a.) Find A_1 by plugging $y = A_1e^t$ into $y'' - 4y' - 5y = e^t$

b.) Find A_2 by plugging $y = A_2te^{-t}$ into $y'' - 4y' - 5y = e^{-t}$

c.) Find A_3, B_3, C_3, D_3 by plugging $y = A_3t^3+B_3t^2+C_3t+D_3$ into $y''-4y'-5y = 2t^3 + 3t^2$

d.) Find A_4, B_4 by plugging $y = A_4sin(3t) + B_4cos(3t)$ into
$y'' - 4y' - 5y = 4sin(3t) + 5cos(t)$

We get the non-homogeneous solution by adding together the non-homogeneous solutions obtained from the above 4 parts since our diff eqn is LINEAR.

We get the general solution by combining the general homogeneous solution with this non-homogeneous solution.

7.) $y'' - 4y' - 5y = e^t + e^{-t} + 2t^3 + 3t^2 + 4sin(3t) + 5cos(t)$

Guess: $\underline{\;y \;=\; (A_1e^t) + (A_2te^{-t}) + (A_3t^3 + B_3t^2 + C_3t + D_3)\;}$

$\underline{+(A_4sin(3t) + B_4cos(3t)) + (A_5sin(t) + B_5cos(t))}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

Guess: $\underline{\;y \;=\; (At^2 + Bt + C)e^{2t}\;}$

Since the RHS is a product, we guess a product.
Note I could have guessed $y = (At^2+Bt+C)De^{2t} = (ADt^2+BDt+CD)e^{2t}$, but since AD, BD, CD are just constants, I don't need D.

Note homogeneous solution to $y'' - 6y' + 9y = 0$ is $y = c_1e^{3t} + c_2te^{3t}$

since $r^2 - 6r + 9 = (r-3)(r-3) = 0$

9.) $y'' - 6y' + 9y = 7e^{3t}$

Guess: $\underline{\quad y \ = \ At^2e^{3t} \quad}$

Note neither $y = Ae^{3t}$ nor $y = Ate^{3t}$ will work since both are homogeneous solutions. But our trick of multiplying by t until we have a guess that is not a homogeneous solution will work.

10.) $y'' - 6y' + 9y = 7e^{-3t}$

Guess: $\underline{\quad y \ = \ Ae^{-3t} \quad}$

$y \ = \ Ae^{-3t}$ is not a homogeneous solution (when $A \neq 0$).

Some special cases:

11.) $y'' - 5y = 4sin(3t)$

Best Guess: $\underline{\quad y \ = \ Asin(3t) \quad}$

Note, we also could have guessed $y = Asin(3t) + Bcos(3t)$, but since there is no y' term, we don't need the cosine term. But both guesses will work. Plugging in $y = Asin(3t) + Bcos(3t)$ will take a little more work, but you will still get the right answer.

12.) $y'' - 4y' = t^2 - 2t + 1$

Guess: $\underline{\quad y \ = \ At^3 + Bt^2 + Ct \quad}$

Note there is no y term on the LHS. Thus to get a t^2 term when we plug in our guess, we will need to plug in a t^3 term. Hence we guess a degree 3 polynomial. Note we don't need to include a constant term; we could have guessed $y \ = \ At^3 + Bt^2 + Ct + D$, but any constant D will work (and hence there are an infinite number of solutions for D) so we might as well take $D = 0$.

Don't worry too much about guessing wrong. You will usually be able to figure out why an incorrect guess doesn't work and use that info to determine a better guess.

2 unknowns ⇒ need 2 eqns

3.5: Solving 2nd order linear non-homogeneous DE using method of undetermined coefficients.

Example: Solve $y'' + 4y = 12t + \boxed{8sin(2t)}$.

Step 1: Solve homogeneous system, $y'' + 4y = 0$

$r^2 + 4 = 0 \quad \Rightarrow \quad r^2 = -4 \quad \Rightarrow \quad r = \pm\sqrt{-4} = 0 \pm 2i$

Hence homogeneous soln is $y = c_1 cos(2t) + c_2 sin(2t)$

Step 2a: Find one solution to $y'' + 4y = 12t$

Possible guess: $y = At + \cancel{B}$. Then $y' = A$ and $y'' = 0$.

Plug in: $0 + 4(At + \cancel{B}) = 12t \Rightarrow 4At + \cancel{4B} = 12t + 0$

Thus $4A = 12$ and $\cancel{4B = 0} \quad \Rightarrow \quad A = 3$ and $B = 0$

Thus $y = 3t$ is a solution to $y'' + 4y = 12t$.

Simpler guess: since there is no y' term, we didn't need the B term in our guess. We could have guessed $y = At$ instead for this particular problem (and other analogous problems). If you make similar observations when you do your HW, you can save time when you do comparable problems.

Step 2b: Find one solution to $y'' + 4y = \boxed{8sin(2t)}$

Incorrect guess: $y = Asin(2t)$. Then $y' = 2Acos(2t)$ and $y'' = -4Asin(2t)$. since homog

Note: since no y' term, did not include a $Bcos(2t)$ term in guess.

0 since hom

Plug in: $-4Asin(2t) + 4Asin(2t) = 8sin(2t)$.

Thus $0 = 8sin(2t)$.

Thus equation has no solution for A. Hence guess is wrong.

Note this guess is wrong because $y = sin(2t)$ is a homogeneous solution. This is why we always solve homogeneous equations first. If a function is a solution to a homogeneous equation, then no constant multiple of that function can be a solution to a non-homogeneous solution since it is a homogeneous solution.

If your normal guess is a homogeneous solution:

$\boxed{\text{Multiply it by } t}$

until it is no longer a homogeneous solution.

General soln: $y = \underbrace{c_1 cos(2t) + c_2 sin(2t)}_{\to 0} + \underbrace{3t}_{\to 12t} \underbrace{\cancel{+ t\,Bsin(2t)}}_{\to 8sin(2t)}$ $-2t\,cos(2t)$?

wrong

product rule

Incorrect guess: $y = Atsin(2t)$.

Then $y' = Asin(2t) + 2Atcos(2t)$ and

$$y'' = 2Acos(2t) + 2Acos(2t) - 4Atsin(2t)$$
$$= 4Acos(2t) - 4Atsin(2t).$$

Plug into $y'' + 4y = 8sin(2t)$:

$$4Acos(2t) - 4Atsin(2t) + 4Atsin(2t) = 8sin(2t)$$

But this equation has no solution for A. Note we need to add a cosine term to our guess so that we can cancel out the cosine term on LHS:

Better guess: $y = t[Asin(2t) + Bcos(2t)]$.

Best guess: $y = Btcos(2t)$

Then $y' = Bcos(2t) - 2Btsin(2t)$

and $y'' = -2Bsin(2t) - 2Bsin(2t) - 4Btcos(2t)$
$= -4Bsin(2t) - 4Btcos(2t)$

Plug into $y'' + 4y = 8sin(2t)$

$$-4Bsin(2t) - 4Btcos(2t) + 4Btcos(2t) = 8sin(2t)$$

$$-4Bsin(2t) = 8sin(2t) \Rightarrow -4B = 8 \Rightarrow B = -2$$

Thus $y = -2tcos(2t)$ is a solution to

$$y'' + 4y = 8sin(2t)$$

Note: Guessing wrong is NOT a big deal. You can use your wrong guess to determine a correct guess (though guessing right the first time will save you time).

Recall you are looking for ONE solution to your NON-homogeneous equation.

- If you find an infinite number of solns, choose one.
- If your guess gives you one solution, use it.
- If your guess leads to no solutions, than make a different (improved) educated guess.

To find general solution to non-homogeneous LINEAR differential equation: combine all solutions

$$y = \underbrace{c_1cos(2t) + c_2sin(2t)}_{\downarrow\, 0} + \underbrace{3t}_{\downarrow\, 12t} - \underbrace{2tcos(2t)}_{\downarrow\, 8sin(2t)}$$

Note that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$

A system of equations is $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

Suppose $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$, and $A\mathbf{p} = \mathbf{b}$, then

$$\begin{aligned} A(c_1\mathbf{u} + c_2\mathbf{v} + \mathbf{p}) &= c_1A\mathbf{u} + c_2A\mathbf{v} + A\mathbf{p} \\ &= c_1(\mathbf{0}) + c_2(\mathbf{0}) + \mathbf{b} = \mathbf{b} \end{aligned}$$

I.e., $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + \mathbf{p}$ is a soln to $A\mathbf{x} = \mathbf{b}$ for any c_1, c_2.

Solve the following systems of equations:

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\3\\0\end{bmatrix} \qquad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2\\5\\8\end{bmatrix}$$

no soln

$$\left[\begin{array}{ccc|ccc}1&2&3&0&0&2\\4&5&6&0&3&5\\7&8&9&0&0&8\end{array}\right]$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\begin{bmatrix}1&2&3&0&0&2\\0&-3&-6&0&3&-3\\0&-6&-12&-7&0&-6\end{bmatrix}$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc}1&2&3&0&0&2\\0&-3&-6&0&3&-3\\0&0&0&0&-6&0\end{array}\right]$$

$\downarrow$ already know sol'n to system b.

$$\left[\begin{array}{ccc|cc}1&2&3&0&2\\0&-3&-6&0&-3\\0&0&0&0&0\end{array}\right]$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix}1&2&3&0&2\\0&1&2&0&1\\0&0&0&0&0\end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \begin{bmatrix}1&0&-1&0&0\\0&1&2&0&1\\0&0&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\-2x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}$$ homog

---

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\3\\0\end{bmatrix}$$ no solution

---

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\5\\8\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underbrace{x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}}_{\text{homog}}+\begin{bmatrix}0\\1\\0\end{bmatrix}$$

homog + 1 nonhomog

Check:
$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\ \&\ \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\5\\8\end{bmatrix}$$

Compare to solving linear homogeneous differential eqn:

Ex: $ay''+by'+cy=g(t)$

1.) Easily solve homogeneous DE: $ay''+by'+cy=0$

$y=e^{rt}\Rightarrow ar^2+br+c=0\Rightarrow y=c_1\phi_1+c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4).

2.) More work: Find one solution to $ay''+by'+cy=g(t)$ (see sections 3.5, 3.6)

If $y=\psi(t)$ is a soln, then general soln to $ay''+by'+cy=g(t)$ is

$$y=c_1\phi_1+c_2\phi_2+\psi$$

Check: $a\phi_1''+b\phi_1'+c\phi_1=0$

$a\phi_2''+b\phi_2'+c\phi_2=0$

$a\psi''+b\psi'+c\psi=g(t)$

To solve $ay''+by'+cy=g_1(t)+g_2(t)$

1.) Solve $ay''+by'+cy=0 \quad\Rightarrow\quad y=c_1\phi_1+c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay''+by'+cy=g_1(t) \quad\Rightarrow\quad y=\psi_1$

2b.) Solve $ay''+by'+cy=g_2(t) \quad\Rightarrow\quad y=\psi_2$

General solution to $ay''+by'+cy=g_1(t)+g_2(t)$ is

$$y=c_1\phi_1+c_2\phi_2+\psi_1+\psi_2$$