$f : A \to B$ is 1:1 iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$\Rightarrow$

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

Hypothesis: $f(x_1) = f(x_2)$. Conclusion $x_1 = x_2$.

Hypothesis implies conclusion.
p implies q.
$$p \Rightarrow q.$$

Note a statement, $p \Rightarrow q$, is true if whenever the hypothesis p holds, then the conclusion q also holds.

To prove that a statement is true:
(1) Assume the hypothesis holds.
(2) Prove the conclusion holds.

Ex: To prove a function is 1:1:
(1) Assume $f(x_1) = f(x_2)$
(2) Do some algebra to prove $x_1 = x_2$.

$[p \Rightarrow q]$ is equivalent to $[\forall p, q \text{ holds}]$.

That is, for everything satisfying the hypothesis p, the conclusion q must hold.

$\sim$ = not

$\forall$ = for all

$\nexists$ = there does not exist

$\exists$ = there exists

A statement is false if the hypothesis holds, but the conclusion need not hold.

Hypothesis does not implies conclusion.
p does not imply q.
$$p \not\Rightarrow q.$$

That is there exists **a specific case** where the hypothesis holds, but the conclusion does not hold.

To prove that a statement is false: Look counter-example

Find an example where the hypothesis holds, but the conclusion does not hold.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

Ex: $f : R \to R$, $f(x) = x^2$ is not 1:1 since $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, but $1 \neq -1$

$\sim [p \Rightarrow q]$ is equivalent to $\sim [\forall p, q \text{ holds}]$.

Thus if $p \Rightarrow q$ is false,
then it is not true that $[\forall p, q \text{ holds}]$.
That is, $\exists p$ such that q does not hold.

Find counter-example st p holds but q does not hold

$p \Rightarrow q$ is false

$f : A \to B$ is 1:1 iff $\underbrace{f(x_1) = f(x_2)}_{\text{hyp}}$ implies $\underbrace{x_1 = x_2}_{\text{cond}}$.

$f : A \to B$ is 1:1 iff $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

$f : A \to B$ is 1:1 iff for all $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

$f : A \to B$ is NOT 1:1 iff there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

$(-1)^2 \neq (1)^2$

Determine if the following functions are 1:1. Prove it.

Goal: $x_1 = x_2$

1.) $f : R \to R$, $f(x) = x^2$

$f(-1) = f(1)$ but $-1 \neq 1$ $\Rightarrow$ not 1:1

2.) $f : [0, \infty) \to R$, $f(x) = x^2$

Suppose $\underbrace{f(x_1) = f(x_2)}_{\text{hyp}}$ $\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm\sqrt{x_2^2}$

$= \pm|x_2|$

3.) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$

$x_1, x_2 \in [0, \infty) \Rightarrow x_1 + x_2 \geq 0$

$x_1 = \pm|x_2| \Rightarrow \underbrace{x_1 = x_2}_{\text{cond}}$

4.) $f : R \to R$, $f(x) = x^3$

5.) $f : R \to R$, $f(x) = 2$

6.) $f : R \to R$, $f(x) = 8x + 2$

7.) $f : R \to R$, $f(x) = x^2 + 3x$

8.) $f : R \to R$, $f(x) = e^x$

9.) $f : R \to R$, $f(x) = x^4 + x^2$

10.) $f : R \to R$, $f(x) = sin(x)$

there does not exist

$f : A \to B$ is onto iff $f(A) = B$.

$f : A \to B$ is onto iff $b \in B$ implies there exists an $a \in A$ such that $f(a) = b$.

$f : A \to B$ is onto iff for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

$f : A \to B$ is NOT onto iff there exists $b \in B$ s. t. there does not exist an $a \in A$ s. t. $f(a) = b$.

Determine if the following functions are onto. If a function is not onto, prove it.

1.) $f : R \to R, f(x) = x^2$

$-5 \in \mathbb{R}$ but $\nexists\, x$ st $f(x) = x^2 = -5 \Rightarrow$ not onto

2.) $f : [0, \infty) \to R,\ f(x) = x^2$

3.) $f : [0, \infty) \to [0, \infty),\ f(x) = x^2$

onto

4.) $f : R \to R,\ f(x) = x^3$

5.) $f : R \to R,\ f(x) = 2$

6.) $f : R \to R,\ f(x) = 8x + 2$

7.) $f : R \to R,\ f(x) = x^2 + 3x$

8.) $f : R \to R,\ f(x) = e^x$

9.) $f : R \to R,\ f(x) = x^4 + x^2$

10.) $f : R \to R,\ f(x) = sin(x)$

bijection $\quad x \rightleftarrows y$

f is invertible iff f is 1:1 and f is onto.

f is NOT invertible iff f is not 1:1 OR f is not onto.

Determine if the following functions are invertible. If a function is not invertible, state why and determine if you can create an invertible function by changing the co-domain.

proof

not onto so no choice inverse

1.) $f : R \to R,\ f(x) = x^2$

not onto $\Rightarrow$ not invertable

$f : R \to [0,\infty),\ f(x) = x^2$ on inv

Negate $p \Leftrightarrow q_1$ and q_2

2.) $f : [0, \infty) \to R,\ f(x) = x^2$

$\sim p \Leftrightarrow \sim q_1$ or $\sim q_2$

3.) $f : [0, \infty) \to [0, \infty),\ f(x) = x^2$

$$\boxed{\begin{array}{c} f : A \to f(A) \\ \text{onto} \end{array}}$$

4.) $f : R \to R,\ f(x) = x^3$

5.) $f : R \to R,\ f(x) = 2$

6.) $f : R \to R,\ f(x) = 8x + 2$

7.) $f : R \to R,\ f(x) = x^2 + 3x$

8.) $f : R \to R,\ f(x) = e^x$

9.) $f : R \to R,\ f(x) = x^4 + x^2$

10.) $f : R \to R,\ f(x) = sin(x)$

not 1:1 $\Rightarrow$ not invertible

Unique representation of $\vec{X}$ in terms of basis

Linear algebra pre-requisites you must know.

$\mathbf{b_1}, \ldots, \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + \ldots + d_n\mathbf{b_n}$
implies $c_1 = d_1,\ c_2 = d_2 \ldots,\ c_n = d_n$.

or equivalently,

$\mathbf{b_1}, \ldots, \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = \ldots c_n$.

Example 1: $\mathbf{b_1} = (1,0,0), \mathbf{b_2} = (0,1,0), \mathbf{b_3} = (0,0,1)$.$\blacksquare$

$$(1;2,3) \neq (1,2,4).$$

If $(a,b,c) = (1,2,3)$ then $a = 1,\ b = 2,\ c = 3$.

Example 2: $\mathbf{b_1} = 1,\ \mathbf{b_2} = t,\ \mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$ then $a = 1,\ b = 2,\ c = 3$.

factored over $\mathbb{R}$

Application: Partial Fractions

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$ 2 degree les

If you don't like denominators, get rid of them:

$$4 = (Ax + B)(x - 3) + C(x^2 + 1)$$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A + C)x^2 + (B - 3A)x - 3B + C$$

I.e., $0x^2 + 0x + 4 = (A + C)x^2 + (B - 3A)x - 3B + C$

Thus $0 = A + C, \quad 0 = B - 3A, \quad 4 = -3B + C.$

$C = -A,\ \ B = 3A,\ \ 4 = -3(3A) + -A \Rightarrow 4 = -10A.$

Hence $A = -\frac{2}{5},\ B = 3(-\frac{2}{5}) = -\frac{6}{5},\ C = \frac{2}{5}$.

$$\text{Thus, } \frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x - \frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$$

$$= \frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}$$

Alternatively, can plug in $x = 3$ to quickly find C and then solve for A and B. Can also use matrices to solve linear eqns.

$4 = (\ )(0) + C(10)$

$\Rightarrow C = \frac{4}{10} = \frac{2}{5}$

FYI

2.3: Modeling with differential equations.

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t)$ = amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t)$ = amount of salt in tank in grams.

Note $Q(0) = 5$ g

$$\text{rate in} = (2 \text{ liters/min})(Q(t) \cdot tsin(t^2) \text{ g/liters}) = 2Qtsin(t^2) \text{ g/min}$$

$$\text{rate out} = (2 \text{ liters/min})\left(\frac{Q(t)g}{4\text{liters}}\right) = \frac{Q}{2} \text{ g/min}$$

$$\frac{dQ}{dt} = \text{rate in - rate out} = 2Qtsin(t^2) - \frac{Q}{2}$$

$$\frac{dQ}{dt} = Q(2tsin(t^2) - \tfrac{1}{2}), \qquad Q(0) = 5$$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

$$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \tfrac{1}{2})dt = \int 2tsin(t^2)dt - \int \tfrac{1}{2}dt$$

Let $u = t^2$, $du = 2tdt$

$$ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C$$
$$= -cos(t^2) - \frac{t}{2} + C$$

$$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$$

$$Q = Ce^{-cos(t^2) - \frac{t}{2}}$$

$Q(0) = 5: \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

Thus $Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$

Long-term behaviour:

$$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.

2.3: Modeling with differential equations.

Ex.: $F = ma = mv'$

$a =$ acceleration $= v' = x''$
$v =$ velocity $= x'$
$x =$ position
$m =$ mass $\qquad mg =$ weight

Model 1: Falling ball near earth, neglect air resistance.

$F_g =$ Gravitational force $= -mg$
IF the positive direction points up.
Note in some examples in the book, the positive direction points down ($F_g = +mg$) while in other examples in the book, the positive direction points up ($F_g = -mg$)

$mv' = -mg$ implies $v' = -g$. Thus $v = -gt + C$.

IVP: $v(0) = v_0$ implies $v_0 = -g(0) + C$ implies $C = v_0$. Thus $v = -gt + v_0$

$x' = v = -gt + v_0$ implies $x = -\frac{1}{2}gt^2 + v_0 t + C$.

IVP: $x(0) = x_0$ implies $x_0 = -\frac{1}{2}g(0)^2 + v_0(0) + C$ implies $C = x_0$.

Thus $x = -\frac{1}{2}gt^2 + v_0 t + x_0$.

Calc 1

Note when ball reaches maximum height $v = 0$

Model 2: Falling ball near earth, include air resistance.

Let $A(v) =$ the force due to air resistance.

$mv' = F_g + R(v) = -mg + A(v) = \frac{dv}{dt}$

Model 3: Far from earth.

$F_g = -mg\frac{R^2}{(R+x)^2}$ where $R =$ radius of the earth.

If x is small, $\frac{R^2}{(R+x)^2} \sim 1$ and thus $F_g = -mg$ when close to earth.

For large x, $mv' = -mg\frac{R^2}{(R+x)^2}$ where R constant.

$\frac{dv}{dt} = -mg\frac{R^2}{(R+x)^2}$ with 3 variables: v, t, x

To eliminate one variable: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

Note this trick can also be used to simplify some problems.