Quiz 3 Form B
Sept 25, 2017

1. Sketch the direction field for the autonomous equation $y' = y^2 - 2y - 8$. Find the equilibrium solutions, and classify them as stable or unstable. Sketch the solution with initial value $y(0) = 1$.

$y^2 - 2y - 8 = (y+4)(y-2) = 0$ implies $y = 4$ and $y = -2$ are equil solns.

[2] Equilibrium solution: $\underline{\; y = 4 \;}$. Stability of this equilibrium solution $\underline{\; \textit{unstable} \;}$.

[2] Equilibrium solution: $\underline{\; y = -2 \;}$. Stability of this equilibrium solution $\underline{\; \textit{stable} \;}$.

Note the equilibrium solution is a constant solution, not a number. Thus I took off 0.5pt per problem if you did not include $y =$ (i.e, 4 is incorrect, but the equation $y = 4$ is correct).

I forgot to change the IVP in the following, so I accepted all of the answers below

[2] If $y = \phi(t)$ is the solution to the initial value problem $y' = y^2 - 5y - 6$, $y(0) = 3$, what happens to $y = \phi(t)$ as t goes to $+\infty$?

$y^2 - 5y - 6 = (y-6)(y+1) = 0$ implies $y = 6$ and $y = -1$ are equil solns. $y \to -1$ as $t \to +\infty$

[2] If $y = \phi(t)$ is the solution to the initial value problem $y' = y^2 - 2y - 8$, $y(0) = 5$, what happens to $y = \phi(t)$ as t goes to $+\infty$?

Note $y(0) = 5 > 4$. Thus $y \to +\infty$ as $t \to +\infty$

[2] If $y = \phi(t)$ is the solution to the initial value problem $y' = y^2 - 2y - 8$, $y(0) = 1$, what happens to $y = \phi(t)$ as t goes to $+\infty$?

Note $y(0) = 1 \in (-4, 2)$. Thus $y \to -2$ as $t \to +\infty$

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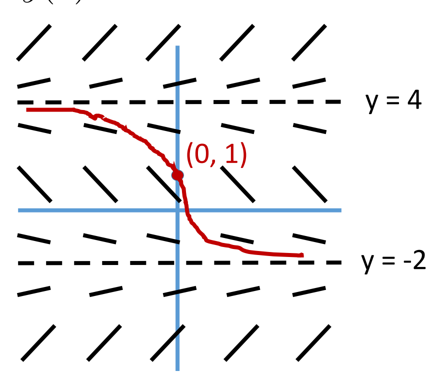
[4] Sketch of Direction field and solution with initial value $y(0) = 3$:

Note one really needs to draw slopes at 9 different y-values (Two for $y > 4$, three for $-2 < y < 4$, and two for $y < -2$ plus the 0-slopes at $y = 4$ and $y = -2$) in order to see all possible solutions.

Note, you don't need to calculate the slopes by plugging in numbers. You just need to know where slope is positive vs negative, small vs large.



Choose the best answer for the following problems:

[2] 2.) If $u = y - x$, then E.) $\frac{dy}{dx} = \frac{du}{dx} + 1$

$y = u + x$ & hence $\frac{dy}{dx} = \frac{d}{dx}(u + x) = \frac{du}{dx} + \frac{dx}{dx} = \frac{du}{dx} + 1$, since derivative is a linear fn.

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[2] 3.) The integrating factor used to solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = x^2$ is

D.) $\frac{1}{x}$ since $u(x) = e^{\int \frac{-1}{x} dx} = e^{-ln|x|} = e^{ln|x|^{-1}} = |x|^{-1}$

Check: If $u(x) = \frac{1}{x} = x^{-1}$, then
$u(x)[\frac{dy}{dx} - \frac{y}{x}] = x^{-1}[y' - x^{-1}y] = x^{-1}y' - x^{-2}y = (x^{-1}y)'$

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[2] 4.) The solution to the initial value problem $y' = \frac{1-x}{y}$, $y(0) = -\sqrt{3}$ is $y = -\sqrt{-x^2 + 2x + 3}$. State the largest interval on which the solution is defined.

$-x^2 + 2x + 3 \geq 0$ and since our convention is to not allow points vertical tangent lines in our domain, $y \neq 0$.

Thus $-x^2 + 2x + 3 = (-x + 3)(x + 1) > 0$. Hence either $(-x + 3)$ and $(x + 1)$ are both positive or both negative. Thus

$-x + 3 > 0$ and $x + 1 > 0$ and thus $3 > x$ and $x > -1$. I.e., $x \in (-1, 3)$ OR
$-x + 3 < 0$ and $x + 1 < 0$ and thus $3 < x$ and $x < -1$. But x can't be both greater than 3 and less than -1, so largest possible domain is B.) (-1, 3).

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[2] 5.) A tank contains 100 liters of pure water. Saline solution with a variable concentration $c(t) = e^{\frac{-t}{100}}$ grams of salt per liter is pumped into the tank at rate 1 liter per minute. The tank is kept perfectly mixed and also drains at a rate of 1 liter per minute, so the volume stays constant. Write an initial value problem modeling $A(t)$, the amount of grams of salt dissolved in the solution in the tank at time t minutes.

Note since the water is pure, the tank starts off with 0 grams of water. Thus $A(0) = 0$.

Rate in $= \frac{e^{\frac{-t}{100}} g}{1L} \frac{1L}{1minute}$ Rate out $= \frac{A(t)g}{1L} \frac{1L}{1minute}$

Thus A.) $\frac{dA}{dt} = e^{\frac{-t}{100}} - \frac{A}{100}$, $A(0) = 0$

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[2] 6.) Note this is a 3.1 problem. The general solution to $y'' - 2y' - 8y = 0$ is

C.) $y = c_1 e^{-2t} + c_2 e^{4t}$ since $r^2 - 2r - 8 = (r - 4)(r + 2) = 0$ implies $r = 4, -2$.