The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$

No external force

2.) Use the fact that the LaPlace Transform is linear:

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

3.) Use thm to change this equation into an algebraic equation:

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$

3.5) Substitute in the initial values:

$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$

Solve for y

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$ simplify

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$s^2 + 3s + 4$: $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an s^2-a^2 or an s^2+a^2 or an $(s-a)^2$, so it must be an $(s-a)^2+b^2$.

Hence we will complete the square:

$s^2 + 3s + \_\_\_ - \_\_\_ + 4 = (s + \_\_\_)^2 - \_\_\_ + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s - a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to use $\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2}) = e^{at}cosbt$
and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2}) = e^{at}sinbt$

$5s + 21 = 5(s + \frac{3}{2}) - \frac{15}{2} + 21 = 5(s + \frac{3}{2}) - \frac{27}{2}$

$= 5(s + \frac{3}{2}) - [\frac{27}{2}\sqrt{\frac{4}{7}}]\sqrt{\frac{7}{4}} = 5(s + \frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5(s+\frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$$= 5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4}) = \mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$$= 5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}) - \frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$$

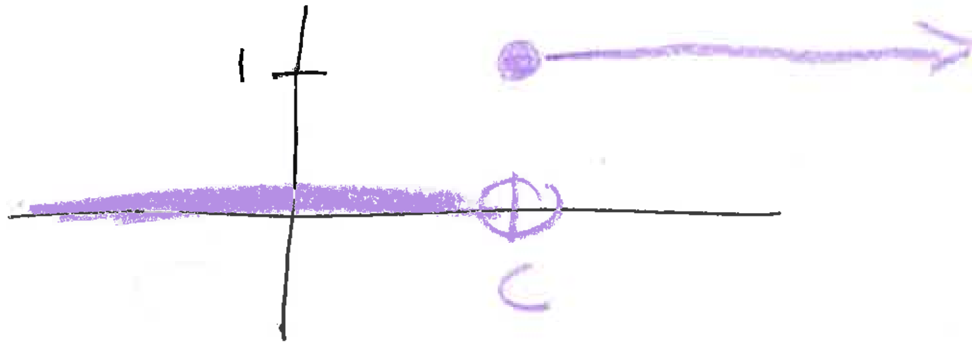
$$= 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$$

Hence $y(t) = 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$.

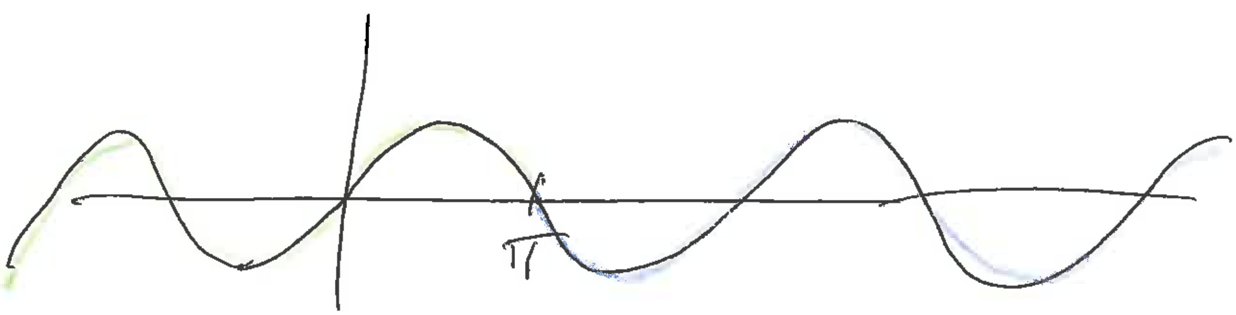
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

6.3: Step functions.

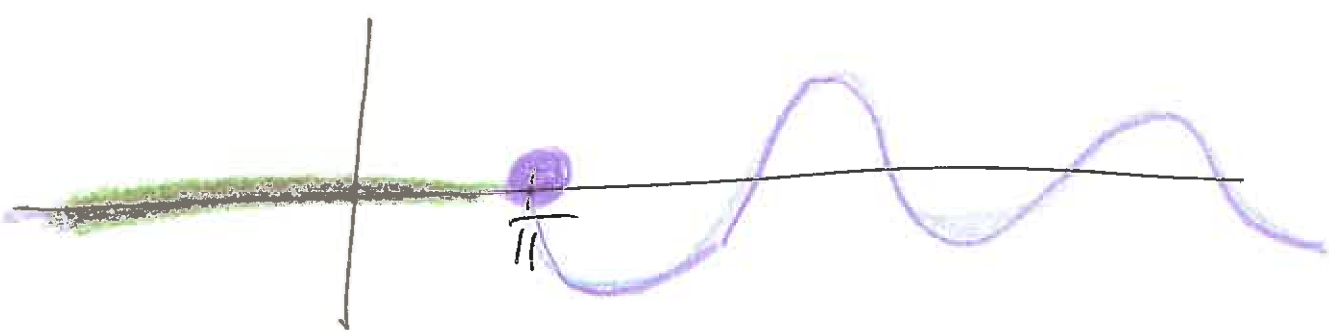
Graph $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$



Graph $g(t) = sin(t)$.



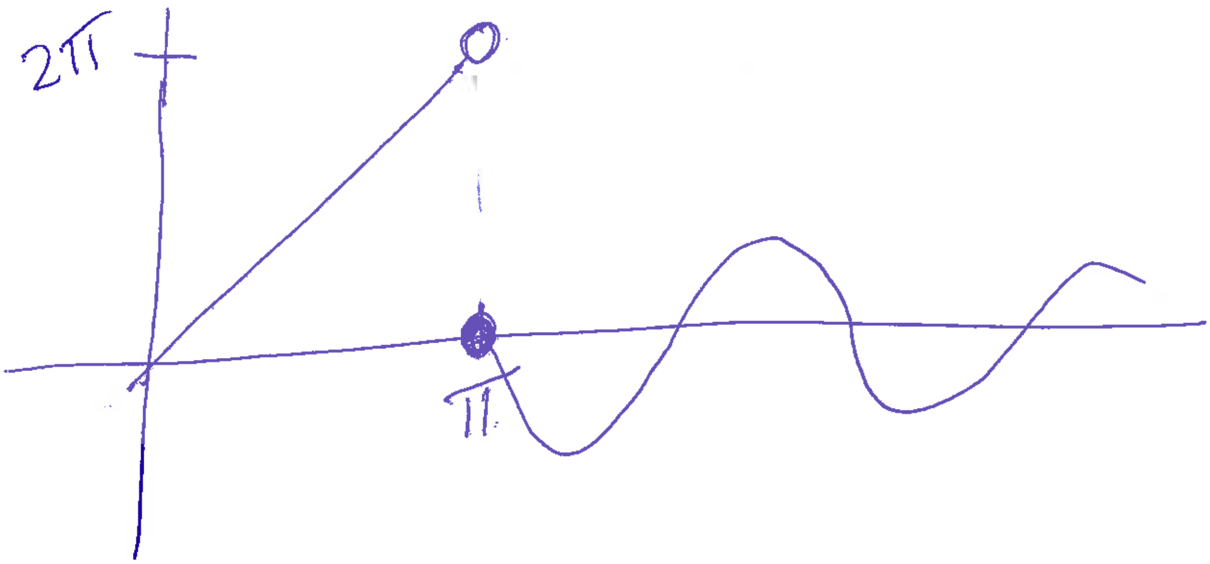
Graph $h(t) = u_\pi(t)sin(t) = \begin{cases} 0 \cdot \sin t = 0 & t < \pi \\ 1 \sin t & t \geq \pi \end{cases}$



Graph $\boxed{f(t) = 2t + u_\pi(t)[sin(t) - 2t]} = \begin{cases} 2t & t < \pi \\ \sin t & t \geq \pi \end{cases}$

$t < \pi$: $f(t) = 2t + 0[\;] = 2t$

$t \geq \pi$: $f(t) = \cancel{2t} + 1(\sin(t) - \cancel{2t}) = \sin(t)$



$$h(t) = \begin{cases} t & 0 \leq t < 4 \\ ln(t) & t \geq 4 \end{cases}$$

implies $h(t) = t + u_4(t)[\ln(t) - t]$

$$j(t) = \begin{cases} t & 0 \le t < 5 \\ 2 & 5 \le t \le 8 \\ e^t & t \ge 8 \end{cases} \quad \text{implies}$$

$$j(t) = t + u_5(t)\left[2 - t\right] + u_8(t)\left[e^t - 2\right]$$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s) = e^{-cs}\mathcal{L}(f(t))$

$\Rightarrow$ Formula 13′

$$\cancel{\mathcal{L}(u_c(t) f(t))}$$

$$\mathcal{L}(u_c(t) \cdot f(t)) = e^{-cs}\mathcal{L}(f(t+c))$$

$$\text{Example: } f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \le t < 5; \\ f_3, & \text{if } 5 \le t < 10; \\ f_4, & \text{if } t \ge 10; \end{cases}$$

Hence

$$f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)]$$
$$+u_{10}(t)[f_4(t) - f_3(t)]$$

Partial check:

$$\text{If } t = 3\text{: } f(3) = f_1(3) + \overset{3<4}{0}\,\cancel{[f_2(3) - f_1(3)]}$$
$$f(3) = f_1(3) \quad +\overset{3<5}{0}\,\cancel{[f_3(3) - f_2(3)]} + \overset{3<10}{0}\,\cancel{[f_4(3) - f_3(3)]} = f_1(3)$$

$$\text{If } t = 9\text{: } f(9) = \cancel{f_1(9)} + \overset{9>4}{1}[\cancel{f_2(9)} - \cancel{f_1(9)}]$$
$$f(9) = f_3(9) \quad +\overset{9>5}{1}[f_3(9) - \cancel{f_2(9)}] + \overset{9<10}{0}\,\cancel{[f_4(9) - f_3(9)]} = f_3(9)$$

Examples:

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t^2 & t \ge 2 \end{cases} \quad \text{implies} \quad \boxed{f(t) = u_2(t)\,t^2}$$

$$f(t) = 0 + u_2(t)\left[t^2 - 0\right]$$

$$g(t) = \begin{cases} t^2 & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases} \quad \text{implies} \quad \boxed{g(t) = t^2 - t^2 u_3(t)}$$

$$g(t) = t^2 + u_3(t)\left[0 - t^2\right]$$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $g(t) = f(t+c)$. Then $g(t-c) = f(t-c+c) = f(t)$. Thus

$$\mathcal{L}(u_c(t)f(t)) = \mathcal{L}(u_c(t)g(t-c)) = e^{-cs}\mathcal{L}(g(t))$$
$$= e^{-cs}\mathcal{L}(f(t+c)).$$

or equivalently

Formula 13'

$$\boxed{\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).}$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Find the LaPlace transform of the following:

a.) $\mathcal{L}(u_3(t)\overbrace{(t^2-2t+1)}^{f(t)}) = e^{-3s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

$$= e^{-3s}\mathcal{L}\left((t+3)^2 - 2(t+3) + 1\right)$$

Formula 13'

$$= e^{-3s}\mathcal{L}(t^2 + 6t + 9 - 2t - 6 + 1)$$

$$= e^{-3s}\mathcal{L}(t^2 + 4t + 4\cdot 1)$$

$$= e^{-3s}\left[\mathcal{L}(t^2) + 4\mathcal{L}(t^1) + 4\mathcal{L}(1)\right] = e^{-3s}\left[\frac{2}{s^3} + 4\left(\frac{1}{s^2}\right) + \frac{4}{s}\right]$$

b.) $\mathcal{L}(u_4(t)(e^{-8t})) =$ $e^{-4s}\mathcal{L}(e^{-8(t+4)})$ See chalkboard notes

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ ________________

Find the LaPlace transform of

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$.

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$

$$\boxed{= \mathcal{L}^{-1}(e^{-cs}\boxed{\mathcal{L}(f(t))}) = u_c(t)f(t-c)}$$

where $f(t) = \mathcal{L}^{-1}(F(s))$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(e^{-8s}\boxed{\frac{1}{s-3}}) =$ $u_8(t)e^{3(t-8)}$

$= u_8(t)\cdot f(t-8)$ where $\mathcal{L}(f(t)) = \dfrac{1}{s-3}$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t} \Rightarrow f(t-8) = e^{3(t-8)}$

b.) $\mathcal{L}^{-1}(e^{-4s}\frac{1}{s^2-3}) =$ $u_4(t)\, f(t-4) = u_4(t)\, \frac{\sinh(\sqrt{3}(t-4))}{\sqrt{3}}$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2-3}\right) = \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2-(\sqrt{3})^2}\right) = \frac{1}{\sqrt{3}} \sinh(\sqrt{3}\, t)$

c.) $\mathcal{L}^{-1}(e^{-s}\frac{5}{(s-3)^4}) =$ $u_1(t) \cdot \left(\frac{5}{6}\right)(t-1)^3 e^{3(t-1)}$

d.) $\mathcal{L}^{-1}(\frac{e^{-s}}{4s}) =$ ____________________

e.) $\mathcal{L}^{-1}(e^{-s}) =$ ____________________

f.) $\mathcal{L}^{-1}(e^{-s}\frac{1}{(s-3)^2+4}) =$ ____________________

g.) $\mathcal{L}^{-1}(e^{-s}\frac{2s-5}{s^2+6s+13}) =$ ____________________