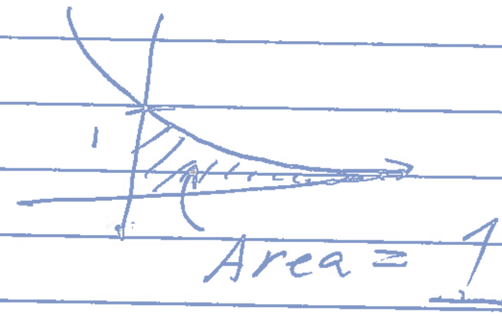
6.1

Improper integrals

$$\int_c^\infty f(t)\,dt = \lim_{A\to\infty}\int_c^A f(t)\,dt$$

Ex $\int_0^\infty e^{-t}\,dt$

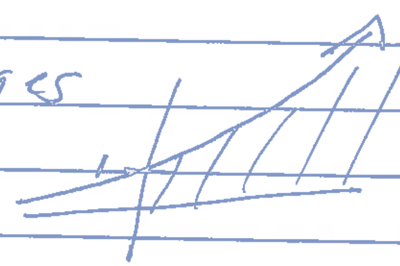


$$= \lim_{A\to\infty}\int_0^A e^{-t}\,dt$$

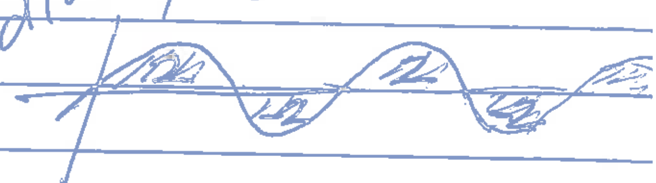
$$= \lim_{A\to\infty}\left(-e^{-t}\right)\Big|_0^A$$

$$= \lim_{A\to\infty}\left(\underbrace{-e^{-A}}_{\to 0} - \left(-e^{-0}\right)\right) = 1$$

Ex: $\int_0^\infty e^{t}\,dt$ diverges



Ex: $\int_0^\infty \sin(t)\,dt$ diverges



$$= \lim_{A\to\infty} -\cos(t)\Big|_0^A = \lim_{A\to\infty}\left[-\cos(A)+1\right]$$

Laplace transform of f

$\mathcal{Q}$:

$$\mathcal{L}(f) = F$$

where $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Ex: $f : \mathbb{R} \to \mathbb{R}$
$f(t) = 1$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt$$

$$= \lim_{A\to\infty} \frac{e^{-st}}{-s}\Big|_0^A$$

$$= \lim_{A\to\infty}\left[\frac{e^{-As}}{-s} - \frac{e^{0}}{-s}\right]$$

$$= 0 + \frac{1}{s} = \frac{1}{s},\ s>0$$

$$\mathcal{L}(1) = \frac{1}{s},\ s > 0$$

Ex: $f(t) = e^{at}$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}(e^{at})\,dt$$

$$= \lim_{A\to\infty} \int_0^A e^{(a-s)t}\,dt$$

$$= \lim_{A\to\infty} \left.\frac{e^{(a-s)t}}{(a-s)}\right|_0^A$$

$$= \lim_{A\to\infty} \left[\frac{e^{(a-s)A}}{a-s} - \frac{e^0}{a-s}\right]$$

$$= 0 - \frac{1}{a-s} \quad \text{If } s > a$$

$$= \frac{1}{s-a}$$

Ex: $\mathcal{L}(\cos(at)) = \int_0^\infty e^{-st}\cos(at)\,dt$

Wed

$$= \mathcal{L}\left(\frac{e^{iat}+e^{-iat}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{iat}+e^{-iat})$$

$$= \frac{1}{2}\left(\frac{1}{s-ia}+\frac{1}{s+ia}\right)$$

$$= \frac{1}{2}\,\frac{s+ia+s-ia}{s^2+a^2}$$

$$= s/(s^2+a^2)$$

: integration by parts
lots of alg
OR use $\cos(at) = \frac{e^{iat}+e^{-iat}}{2}$

$$= \frac{s}{s^2+a^2} \quad \text{if } s > 0$$

$$f(t)=1 \xrightarrow{\mathcal{L}} F(s)=\frac{1}{s}$$

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$$f(t)=e^{at} \xrightarrow{\mathcal{L}} F(s)=\frac{1}{s-a}$$

$$f(t)=\cos(at) \xrightarrow{\mathcal{L}} F(s)=\frac{s}{a^2+s^2}$$

For convenience write

f as a fn of t

$\mathcal{L}(f) = F$ " " s

write large

Note $\mathcal{L}$ is linear

$$\mathcal{L}(cf+dg) = \int_0^\infty e^{-st}[cf+dg]\,dt$$

$$= c\int_0^\infty e^{-st} f\,dt + d\int_0^\infty e^{-st} g\,dt$$

$$= c\mathcal{L}(f) + d\mathcal{L}(g)$$

$$\mathcal{L}(2+3e^{4t}+5e^{-6t}) = 2\mathcal{L}(1)+3\mathcal{L}(e^{4t})+5\mathcal{L}(e^{-6t})$$

$$= \boxed{\frac{1}{s} + \frac{1}{s-4} + \frac{1}{s-(-6)} \quad s>4}$$

where $s>0$ & $s>4$ and $s>-6$ ie