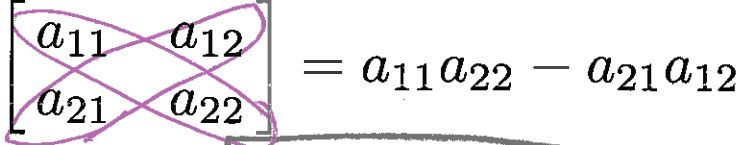


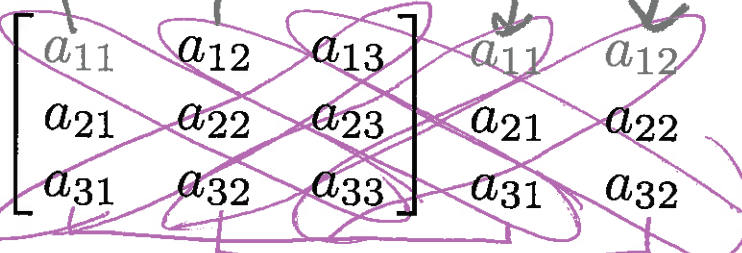
Linear Algebra Review: Determinants

Defn: $\det A = \sum \pm a_{1j_1} a_{2j_2} \dots a_{nj_n}$

2×2 short-cut: $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$



3×3 short-cut: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$



Note there is no short-cut for $n \times n$ matrices when $n > 3$.

Definition of Determinant using cofactor expansion

Defn: A_{ij} is the matrix obtained from A by deleting the i th row and the j th column.

Thm: Let $A = (a_{ij})$ be an $n \times n$ square matrix, $n > 1$. Then expanding along row i ,

$$\det A = \sum_{k=1}^n (-1)^{i+k} a_{ik} \det A_{ik}.$$

Or expanding along column j ,

$$\det A = \sum_{k=1}^n (-1)^{k+j} a_{kj} \det A_{kj}.$$

Properties of Determinants

Thm: If $A \xrightarrow{R_i \rightarrow cR_i} B$, then $\det B = c(\det A)$.

Thm: If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det B = -(\det A)$.

Thm: If $A \xrightarrow{R_i + cR_j \rightarrow R_i} B$, then $\det B = \det A$.

change the determinant

Does not change determinant

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $\det A = a_{11}a_{22}\dots a_{nn}$, the product of the entries along the main diagonal.

Thm: A square matrix is invertible if and only if
 $\det A \neq 0$.

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $\det A \neq 0$.

Linear Algebra Review: Cramer's Rule.

FYI

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with \mathbf{b} .

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $\det A \neq 0$. Then

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}.$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

FYI

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$$

$$\det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$$

$$\det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$$

$$\text{Thus } x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = \frac{9}{2}.$$

3.6 Variation of Parameters

Solve $ay'' + by' + cy = g(t)$

1) Find homogeneous solutions:

Suppose general homogeneous soln to $ay'' + by' + cy = 0$ is

$$y = c_1\phi_1(t) + c_2\phi_2(t) \leftarrow \text{homog}$$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: Guess $y = u_1(t)\phi_1 + u_2(t)\phi_2$ and solve for u_1 and u_2

$$u_1(t) = \int \begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix} \frac{g(t)}{a} dt = - \int \frac{\phi_2(t)g(t)}{aW(\phi_1, \phi_2)} dt$$

$$u_2(t) = \int \begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix} \frac{g(t)}{a} dt = \int \frac{\phi_1(t)g(t)}{aW(\phi_1, \phi_2)} dt$$

Note that in both cases, you divide by the Wronskian:

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

General non-homogeneous solution :

$$y = \underbrace{c_1\phi_1(t) + c_2\phi_2(t)}_{\text{homog}} + \underbrace{(u_1(t)\phi_1 + u_2(t)\phi_2)}_{\text{a non homog}}$$

Note: If you are likely to forget to divide by a when finding u_1 and u_2 , then when finding non-homogeneous solution, divide both sides of DE by a so that you have $1y'' + py' + qy = \frac{g(t)}{a}$ (this is what our textbook does).

Note: These formulas generalize to higher order linear DEs. To find u_i replace the i^{th} column of the Wronskian with a column of all zeros except the last entry which is 1 and follow 2nd order case.

Note: ORDER MATTERS. Once you choose an ordering of your homogeneous solutions ϕ_1, \dots, ϕ_n for an n th order DE, be consistent with that order.

Note: If you are solving an IVP, you must find the complete general solution including any non-homogeneous parts **BEFORE** using your initial values to solve for the c_i 's.

3.6 Variation of Parameters

$$\text{Solve } y'' - 2y' + y = e^t \ln(t)$$

1) Find homogeneous solutions:

$$\text{Solve } y'' - 2y' + y = 0$$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and
 $r^2e^{rt} - 2re^{rt} + e^{rt} = 0$ implies $r^2 - 2r + 1 = 0$

$$(r - 1)^2 = 0, \text{ and hence } r = 1$$

General homogeneous solution: $y = c_1e^t + c_2te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$ ■

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$$\begin{aligned} W(\phi_1, \phi_2) &= \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} \\ &= e^{2t} + te^{2t} - te^{2t} = e^{2t} \end{aligned}$$

$$\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix} = -\phi_2 = -te^t \quad \begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix} = \phi_1 = e^t$$

$$\begin{aligned}
u_1(t) &= \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{W(\phi_1, \phi_2)} \frac{g(t)}{a} dt = - \int \frac{(te^t)(e^t \ln(t))}{e^{2t}} dt \\
&= - \int t \ln(t) dt = - \left[\frac{t^2 \ln(t)}{2} - \int \frac{t}{2} dt \right] = - \frac{t^2 \ln(t)}{2} + \frac{t^2}{4}
\end{aligned}$$

Integration by parts: $u = \ln(t) \quad dv = t dt$
 $du = \frac{dt}{t} \quad v = \frac{t^2}{2}$

$$\begin{aligned}
u_2(t) &= \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t) dt = \int \frac{(e^t)(e^t \ln(t))}{e^{2t}} dt \\
&= \int \ln(t) dt = [t \ln(t) - \int dt] = t \ln(t) - t
\end{aligned}$$

Integration by parts: $u = \ln(t) \quad dv = dt$
 $du = \frac{dt}{t} \quad v = t$

General solution :

$$y = c_1 e^t + c_2 t e^t + \left(-\frac{t^2 \ln(t)}{2} + \frac{t^2}{4} \right) e^t + (t \ln(t) - t) t e^t$$

which simplifies to $y = c_1 e^t + c_2 t e^t + \left(\frac{\ln(t)}{2} - \frac{3}{4} \right) t^2 e^t$