Solve $y''' + y'' + 3y' + 10y = 0$

Guess $y = e^{rt}$

Solving polynomial equations:

Example: $r^3 + r^2 + 3r + 10 = 0$

Plug in $r = \pm 1, \pm 2, \pm 5, \pm 10$ to see if any of these are solns:

$(\pm 1)^3 + (\pm 1)^2 + 3(\pm 1) + 10 \neq 0$

$(\pm 2)^3 + (\pm 2)^2 + 3(\pm 2) + 10 \; ? = ?0$

$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$ ✓

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + 3r + 10$

Hence $r^3 + 1r^2 + 3r + \boxed{10} = (r + \boxed{2})(r^2 + \underline{-1}\, r + \boxed{5})$ $\quad 5 = \frac{10}{2}$

$1r^2 = 2r^2 + -1r^2$

To find the coefficient of r in the above, you can do so by (1) long division, (2) inspection, (3) using variable x

$r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \underline{\;x\;}\, r + 5)$

$(r+2)(r^2 + \underline{\;x\;}\, r + 5) = r^3 + (2+x)r^2 + (2x+5)r + 10$

$r^3 \quad + \quad r^2 \quad + \quad 3r \quad + \quad 10$

Thus $2 + x = 1$ and hence $x = -1$

Check: $2x + 5 = 2(-1) + 5 = 3$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$. $\quad$ Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$. ← Guessed $y = e^{rt}$

$y = c_1\left(e^{-2t}\right) + c_2\left(e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right)\right) + c_3\left(e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)$

$y^{(n)} + b = 0$

In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies

$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$

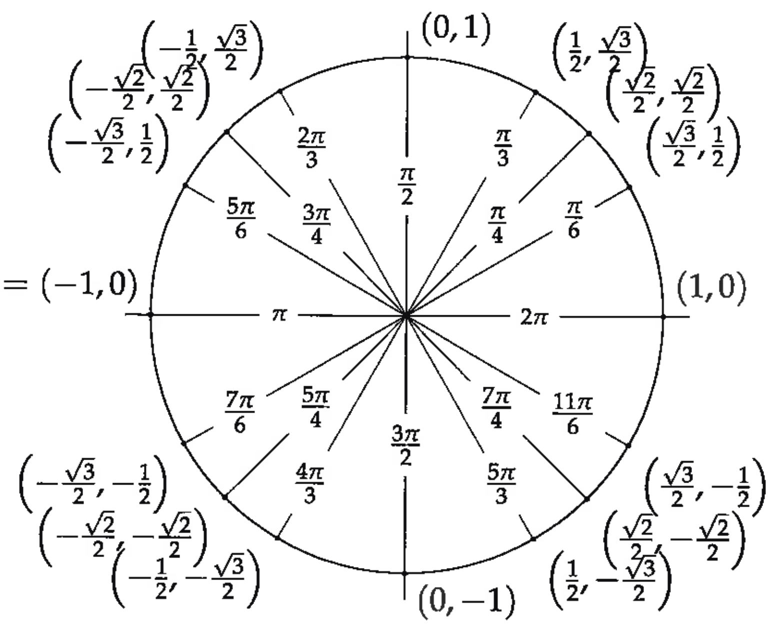
$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\frac{i\pi}{4}) + isin(\frac{i\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\frac{3i\pi}{4}) + isin(\frac{3i\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\frac{5i\pi}{4}) + isin(\frac{5i\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\frac{7i\pi}{4}) + isin(\frac{7i\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$-1 + 0i = (-1, 0)$

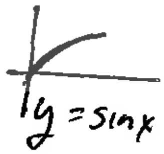


$sin(0) = \frac{\sqrt{0}}{2} = 0$

$sin(\frac{\pi}{6}) = \frac{\sqrt{1}}{2} = \frac{1}{2}$

$sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$



$sin(\frac{\pi}{2}) = \frac{\sqrt{4}}{2} = 1$

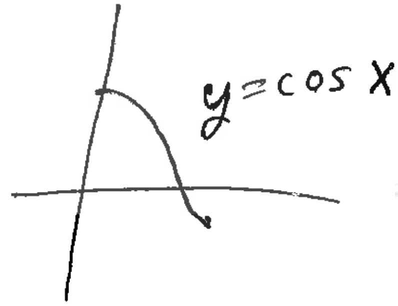
$cos(0) = 1$

$cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$cos(\frac{\pi}{3}) = \frac{1}{2}$

$cos(\frac{\pi}{2}) = 0$



Example: Solve $y^{(iv)} + y = 0$

$y = e^{rt}$ implies $r^4 + 1 = 0$ and thus

$$r = \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \text{ and } r = -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$$

Thus general homogeneous solution is

$$y = c_1 e^{\frac{\sqrt{2}}{2}t} cos(\tfrac{\sqrt{2}}{2}t) + c_2 e^{\frac{\sqrt{2}}{2}t} sin(\tfrac{\sqrt{2}}{2}t)$$
$$+ c_3 e^{-\frac{\sqrt{2}}{2}t} cos(\tfrac{\sqrt{2}}{2}t) + c_4 e^{-\frac{\sqrt{2}}{2}t} sin(\tfrac{\sqrt{2}}{2}t)$$

From Wednesday

Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$\text{IVP: } u(t_0) = u_0, \quad , u'(t_0) = u_1$$

NOTE: Positive direction points DOWN.

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

g = 9.8 m/sec^2 or 32 ft/sec^2.

Weight $= mg \qquad mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$

A mass of 3kg stretches a spring 4.9 m. If the mass is acted upon by an external force of $40e^{\frac{-t}{3}}$ N in a medium that imparts a viscous force of 10 N when the speed of the mass is 5 m/sec. If the mass is pulled down 1 m and set in motion with an upward velocity of 8 m/sec, describe the motion of the mass.

$m = 3$

$|F_{damping}(t)| = |\gamma u'(t)| \Rightarrow 10 = \gamma(5)$. Thus $\gamma = 2$

$mg - kL = 0$ implies $= \frac{mg}{L} = \frac{3(9.8)}{4.9} = 6$

$$\boxed{\text{IVP: } 3u''+2u'+6u = 40e^{\frac{-t}{3}}, \quad u(0) = 1, \quad u'(0) = -8}$$

IVP: $3u''+2u'+6u = 40e^{\frac{-t}{3}}, \quad u(0) = 1, \quad u'(0) = -8$

Step 1: Solve homogeneous eqn: $3u'' + 2u' + 6u = 0$

$u = e^{rt}$ implies $3r^2 + 2r + 6 = 0$ implies

$$r = \frac{-2 \pm \sqrt{4 - 4(3)(6)}}{2(3)} = \frac{-2}{2(3)} \pm \frac{\sqrt{4}\sqrt{1-(3)(6)}}{2(3)} = -\frac{1}{3} \pm i\frac{\sqrt{17}}{3}$$

Thus general homogeneous solution is

$$u(t) = c_1 e^{-\frac{t}{3}} cos(\tfrac{\sqrt{17}}{3}t) + c_2 e^{-\frac{t}{3}} sin(\tfrac{\sqrt{17}}{3}t)$$

Step 2: Find a non-homogeneous soln

Section 3.6: First find Wronskian,

$$W(e^{-\frac{t}{3}} cos(\tfrac{\sqrt{17}}{3}t), e^{-\frac{t}{3}} sin(\tfrac{\sqrt{17}}{3}t)).$$

Too much work. Thus,

Section 3.5: Guess $u(t) = Ae^{\frac{-t}{3}}$.

Then $u'(t) = -\frac{A}{3}e^{\frac{-t}{3}}$ and $u''(t) = \frac{A}{9}e^{\frac{-t}{3}}$

$3(\frac{A}{9}e^{\frac{-t}{3}}) + 2(-\frac{A}{3}e^{\frac{-t}{3}}) + 6Ae^{\frac{-t}{3}} = 40e^{\frac{-t}{3}}$

$\frac{A}{3} - \frac{2A}{3} + 6A = 40$ implies $A - 2A + 18A = 17A = 120$.

Thus $A = \frac{120}{17}$ and

hence $u(t) = \frac{120}{17}e^{\frac{-t}{3}}$ is a non-homogeneous soln.

Thus general NON-homogeneous solution is

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t)] + \frac{120}{17}e^{\frac{-t}{3}}$$

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

IF IVP 3.7, 3.8

Step 3: Use initial values to find c_1 and c_2.

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$u'(t) = -\frac{1}{3}e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$+e^{-\frac{t}{3}}[-c_1\frac{\sqrt{17}}{3}sin(\frac{\sqrt{17}}{3}t) + c_2\frac{\sqrt{17}}{3}cos(\frac{\sqrt{17}}{3}t)]$$

$u(0) = 1$: $\quad 1 = c_1(1) + c_2(0) + \frac{120}{17}$

implies $c_1 = 1 - \frac{120}{17} = -\frac{103}{17}$.

$u'(0) = -8$: $\quad -8 = -\frac{1}{3}[c_1 + \frac{120}{17}] + c_2\frac{\sqrt{17}}{3}$

$$-24 = -[-\frac{103}{17} + \frac{120}{17}] + c_2\sqrt{17}$$

$-24 = -[\frac{17}{17}] + c_2\sqrt{17}$ implies $c_2\sqrt{17} = -23$ and thus $c_2 = -\frac{23}{\sqrt{17}} = -\frac{23\sqrt{17}}{17}$

Thus solution to IVP is

$$u(t) = e^{-\frac{t}{3}}[-\frac{103}{17}cos(\frac{\sqrt{17}}{3}t) - \frac{23\sqrt{17}}{17}sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

For word problems

Step 4: continue simplifying $\Rightarrow R\cos(\omega t-\delta)$

Thus solution to IVP is

$$u(t) = e^{-\frac{t}{3}}[-\frac{103}{17}cos(\frac{\sqrt{17}}{3}t) - \frac{23\sqrt{17}}{17}sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

Simplify:

$$u(t) = e^{-\frac{t}{3}}[Rcos\delta cos(\frac{\sqrt{17}}{3}t) + Rsin\delta sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$u(t) = e^{-\frac{t}{3}}[Rcos(\frac{\sqrt{17}}{3}t - \delta) + \frac{120}{17}]$$

where $Rcos\delta = c_1 = -\frac{103}{17}$ and $Rsin\delta = c_2 = -\frac{23\sqrt{17}}{17}$.

Thus $R = \sqrt{R^2cos^2\delta + R^2sin^2\delta} = \sqrt{c_1^2 + c_2^2}$

$$R = \sqrt{(-\frac{103}{17})^2 + (-\frac{23\sqrt{17}}{17})^2} = \frac{\sqrt{19602}}{17} = \frac{\sqrt{2(3)^4(11)^2}}{17} = \frac{99\sqrt{2}}{17}$$

and $\frac{c_2}{c_1} = \frac{Rsin\delta}{Rcos\delta} = tan\delta$. Thus $\delta = tan^{-1}(\frac{c_2}{c_1})$, sort of – you must choose the correct quadrant based on the signs of c_1 and c_2.

$$\delta = tan^{-1}(\frac{23\sqrt{17}}{103}) + \pi \sim 222.64^o \sim 3.8858 \text{ radians}$$
$$\sim 1.24\pi \text{ radians} \sim \frac{5\pi}{4} \text{ radians.}$$

Simplified answer to IVP:

$$u(t) = e^{-\frac{t}{3}}[\frac{99\sqrt{2}}{17}cos(\frac{\sqrt{17}}{3}t - (tan^{-1}(\frac{23\sqrt{17}}{103}) + \pi)) + \frac{120}{17}]$$

Approximation of solution:

$$u(t) \sim e^{-\frac{t}{3}}[\frac{99\sqrt{2}}{17}cos(\frac{\sqrt{17}}{3}t - \frac{5\pi}{4}) + \frac{120}{17}]$$