Summary of sections 3.1, 3, 4: Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = \boxed{e^{rt}}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $\boxed{ar^2 + br + c = 0,}$

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

3.1) 2 real solns

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

3.3) 2 complex solns

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

3.4) 1 repeated root

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for c_1, c_2 to find unique solution.

First find complete sol'n to entire problem

Examples:

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$r^2e^{rt} - 3re^{rt} - 4e^{rt} = 0$

$r^2 - 3r - 4 = 0$ implies $(r-4)(r+1) = 0$. Thus $r = 4, -1$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP:
Need to solve for 2 unknowns, c_1 & c_2; thus need 2 eqns:

$y = c_1e^{4t} + c_2e^{-t}$, $\quad y(0) = 1$ implies $1 = c_1 + c_2$

$y' = 4c_1e^{4t} - c_2e^{-t}$, $\quad y'(0) = 2$ implies $2 = 4c_1 - c_2$

Thus $3 = 5c_1$ & hence $c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

Thus IVP soln: $y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$

Ex 2: Solve $y'' - 3y' + 4y = 0$.

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1e^{\frac{3}{2}t}cos(\frac{\sqrt{7}}{2}t) + c_2e^{\frac{3}{2}t}sin(\frac{\sqrt{7}}{2}t)$

Ex 3: $y'' - 6y' + 9y = 0$ implies $r^2 - 6r + 9 = (r-3)^2 = 0$
Repeated root, $r = 3$ implies
general solution is $y = c_1e^{3t} + c_2te^{3t}$

Back to our class

Compare to solving linear homogeneous differential eqn:

Ex: $ay'' + by' + cy = g(t)$ For ch 3 & 4 step 1

Step 1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4).

Step 2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections 3.5, 3.6) Step 2 for non homog

If $y = \psi(t)$ is a soln, then general soln to $ay''+by'+cy = g(t)$ is

$$\boxed{y = c_1\phi_1 + c_2\phi_2 + \psi}$$

Check: $a\phi_1'' + b\phi_1' + c\phi_1 = 0$

$a\phi_2'' + b\phi_2' + c\phi_2 = 0$

$a\psi'' + b\psi' + c\psi = g(t)$

To solve $ay'' + by' + cy = g_1(t) + g_2(t)$

1.) Solve $ay'' + by' + cy = 0 \quad \Rightarrow \quad y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \quad \Rightarrow \quad y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \quad \Rightarrow \quad y = \psi_2$

Divide problem into simpler parts

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2$$

$0 + g_1(t) + g_2(t)$

If plug into LHS ⟹ RHS

IVP plug in initial values to find c_1 & c_2 to final general soln

Monday's Chalk board example

3.5: Solving 2nd order linear non-homogeneous DE using method of undetermined coefficients.

2a 2b

Example: Solve $y'' + 4y = 12t + 8sin(2t)$.

Step 1: Solve homogeneous system, $y'' + 4y = 0$

$r^2 + 4 = 0 \quad \Rightarrow \quad r^2 = -4 \quad \Rightarrow \quad r = \pm\sqrt{-4} = 0 \pm 2i$

Hence homogeneous soln is $y = c_1 cos(2t) + c_2 sin(2t)$

$\boxed{\text{Step 2a:}}$ Find one solution to $y'' + 4y = 12t$

Possible guess: $y = At + B$. Then $y' = A$ and $y'' = 0$.

Plug in: $0 + 4(At + B) = 12t \Rightarrow 4At + 4B = 12t + 0$

Thus $4A = 12$ and $4B = 0 \quad \Rightarrow \quad A = 3$ and $B = 0$

Thus $y = 3t$ is a solution to $y'' + 4y = 12t$.

Simpler guess: since there is no y' term, we didn't need the B term in our guess. We could have guessed $y = At$ instead for this particular problem (and other analogous problems). If you make similar observations when you do your HW, you can save time when you do comparable problems.

Step 2b: Find one solution to $y'' + 4y = 8sin(2t)$

Incorrect guess: $y = Asin(2t)$. Then $y' = 2Acos(2t)$ and $y'' = -4Asin(2t)$.

Note: since no y' term, did not include a $Bcos(2t)$ term in guess.

y = A sin(2t) is homog soln

Plug in: $\underbrace{-4Asin(2t) + 4Asin(2t)}_{} = 8sin(2t)$.

$$\text{Thus } 0 = 8sin(2t).$$

Thus equation has no solution for A. Hence guess is wrong.

Note this guess is wrong because $y = sin(2t)$ is a homogeneous solution. This is why we always solve homogeneous equations first. If a function is a solution to a homogeneous equation, then no constant multiple of that function can be a solution to a non-homogeneous solution since it is a homogeneous solution.

If your normal guess is a homogeneous solution:

Multiply it by t

until it is no longer a homogeneous solution.

Incorrect guess: $y = Atsin(2t)$.

Then $y' = Asin(2t) + 2Atcos(2t)$ and

$$\begin{aligned} y'' &= 2Acos(2t) + 2Acos(2t) - 4Atsin(2t) \\ &= 4Acos(2t) - 4Atsin(2t). \end{aligned}$$

Plug into $y'' + 4y = 8sin(2t)$:

$$4Acos(2t) - 4Atsin(2t) + 4Atsin(2t) = 8sin(2t)$$

But this equation has no solution for A. Note we need to add a cosine term to our guess so that we can cancel out the cosine term on LHS:

Better guess: $y = t[Asin(2t) + Bcos(2t)]$.

Best guess: $y = Btcos(2t)$

Then $y' = Bcos(2t) - 2Btsin(2t)$

$$\begin{aligned} \text{and } y'' &= -2Bsin(2t) - 2Bsin(2t) - 4Btcos(2t) \\ &= -4Bsin(2t) - 4Btcos(2t) \end{aligned}$$

Plug into $y'' + 4y = 8sin(2t)$

$$-4Bsin(2t) - 4Btcos(2t) + 4Btcos(2t) = 8sin(2t)$$

$$-4Bsin(2t) = 8sin(2t) \;\Rightarrow\; -4B = 8 \;\Rightarrow\; B = -2$$

Thus $y = -2tcos(2t)$ is a solution to

$$y'' + 4y = 8sin(2t)$$

Note: Guessing wrong is NOT a big deal. You can use your wrong guess to determine a correct guess (though guessing right the first time will save you time).

Recall you are looking for ONE solution to your NON-homogeneous equation.

- If you find an infinite number of solns, choose one.
- If your guess gives you one solution, use it.
- If your guess leads to no solutions, than make a different (improved) educated guess.

To find general solution to non-homogeneous LINEAR differential equation: combine all solutions

$$y = \underbrace{c_1cos(2t) + c_2sin(2t)}_{0} + \underbrace{3t}_{12t} - \underbrace{2tcos(2t)}_{8sin(2t)}$$

plug into LHS

⇓

RHS $=$ 0 $+$ $12t$ $+$ $8sin(2t)$

Guess a possible non-homog soln for the following DEs:

Step 1 Note homogeneous solution to $y'' - 4y' - 5y = 0$ is
$y = c_1 e^{-t} + c_2 e^{5t}$ since $r^2 - 4r - 5 = (r-5)(r+1) = 0$

1.) $y'' - 4y' - 5y = 4e^{2t}$

Guess: $y = Ae^{2t}$

2.) $y'' - 4y' - 5y = t^2 - 2t + 1$

Same for $y'' - 4y' - 5y = t^2$

Guess: $y = At^2 + Bt + C$

If have degree 2 polynom $\Rightarrow$ guess is degree 2 polynomial for this problem

3.) $y'' - 4y' - 5y = 4sin(3t)$

Guess: $y = A\cos(3t) + B\sin(3t)$

same guess

4.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

Guess: $y = A\sin(3t) + B\cos(3t)$

5.) $y'' - 4y' - 5y = 4e^{-t}$

Guess: $y = Ate^{-t}$

Note need t since $y = e^{-t}$ is a homog soln

Part 1 Solve homog Part 2

6.) $y'' - 4y' - 5y = \underbrace{e^t}_{a} + \underbrace{e^{-t}}_{b} + \underbrace{2t^3 + 3t^2}_{c} + \underbrace{4sin(3t) + 5cos(3t)}_{d}$

Guess: $y = \underbrace{(A_1 e^t)}_{\text{parts } a} + \underbrace{(A_2 te^{-t})}_{b} + (A_3 t^3 + B_3 t^2 + C_3 t + D_3) + (A_4 \sin(3t) + B_4 \cos(3t))$

7.) $y'' - 4y' - 5y = e^t + e^{-t} + 2t^3 + 3t^2 + 4sin(3t) + 5cos(t)$

Guess: $y = (Ae^t) + (Be^{-t}t) + (Ct^3 + Dt^2 + Et + F)$
$+ (A\sin 3t + B\cos(3t)) + (A\sin t + B\cos t)$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

Guess: $y = (At^2 + Bt + C) \cdot e^{2t}$

Note homogeneous solution to $\boxed{y'' - 6y' + 9y = 0}$ is repeated root
$y = c_1 e^{3t} + c_2 te^{3t}$ since $r^2 - 6r + 9 = (r-3)(r-3) = 0$

9.) $y'' - 6y' + 9y = 7e^{3t}$

Guess: $y = At^2 e^{3t}$

since $y = e^{3t}$ & $y = te^{3t}$ are homog solns

10.) $y'' - 6y' + 9y = 7e^{-3t}$

Guess: $y = Ae^{-3t}$

Some special cases:

plus in 1 term no y' $\Rightarrow$ no cosine term

no y' term

11.) $y'' - 5y = 4sin(3t)$

Best Guess: $y = A\sin(3t) + \cancel{B\cos(3t)}$

12.) $y'' - 4y' = t^2 - 2t + 1$

any D will work so take $D = 0$ so don't need $+D$

Guess: $y = At^3 + Bt^2 + Ct + \cancel{D}$

no y-term $\Rightarrow$ need t^3 term to get y^2 term out

Thm: Suppose $\overbrace{c_1\phi_1(t)+c_2\phi_2(t)}^{\text{homog}}$ is a general solution to

$$ay''+by'+cy=0,$$

If ψ is a solution to

non hom $$ay''+by'+cy=g(t)\ [*],$$

Then $\boxed{\underbrace{\psi}_{\text{nonh}}+\underbrace{c_1\phi_1(t)+c_2\phi_2(t)}_{\text{homog}}}$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\boxed{\gamma=\underbrace{\psi}_{\text{nonhom}}+\underbrace{c_1\phi_1(t)+c_2\phi_2(t)}_{\text{homog}}}$$

Or in other words, $\psi+c_1\phi_1(t)+c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f)=af''+bf'+cf$.

Recall L is a linear function.

Let $\overset{\text{homog soln}}{h}=c_1\phi_1(t)+c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay''+by'+cy=0$,

$$ah''+bh'+ch=0$$

Since ψ is a solution to $ay''+by'+cy=g(t)$,

$$a\psi''+b\psi'+c\psi=g$$

We will now show that $\psi + \overbrace{(c_1\phi_1(t) + c_2\phi_2(t))}^{h} = \psi + h$ is also a solution to [*].

$a(\psi+h)''+b(\psi+h)'+c(\psi+h)$

$(a\psi''+b\psi'+c\psi)+(ah''+bh'+ch)$

$= g + 0 = g$

Since γ a solution to $ay'' + by' + cy = g(t)$,

$a\gamma''+b\gamma'+c\gamma = g$

$a\psi''+b\psi'+c\psi = g$

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

$a(\gamma-\psi)''+b(\gamma-\psi)'+c(\gamma-\psi)$

$= g - g = 0$

Since $\underline{\gamma - \psi}$ is a solution to $ay'' + by' + cy = \underline{0}$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = \underline{\quad c_1\phi_1 + c_2\phi_2 \quad}$$

nonhog homog

Thus $\gamma = \psi + \underbrace{c_1\phi_1(t) + c_2\phi_2(t)}$.

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

Step 1.) Find the general solution to $ay'' + by' + cy = 0$:

solve homog $$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution ψ_i into $ay'' + by' + cy = g_i$.

The general solution to [**] is

$$\underbrace{c_1\phi_1 + c_2\phi_2}_{0} + \underset{g_1}{\underset{\downarrow}{\psi_1}} + \underset{g_2}{\underset{\downarrow}{\psi_2}} + \underset{g_n}{\underset{\downarrow}{...\psi_n}}$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

can break into simpler problems