Note: You must be able to identify which techniques you need to use. For example:

Integration:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.

For differential equations:

Is the differential equation 1rst order or 2nd order?

If 2nd order: Section 3.1, solve $ay'' + by' + cy = 0$.

Guess $y = e^{rt}$.

characteristic eqn

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Need to have two independent solutions.

If $y = \phi_1, y = \phi_2$ are solutions to a LINEAR HOMOGENEC differential equation, $y = c_1\phi_1 + c_2\phi_2$ is also a solution

If 1st order: Is the equation linear or separable or ? or bernoulli

IVP ⇐ plug in to find constants

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

Check

$$\begin{aligned} y' + py &= g \\ y'u + upy &= ug \\ (uy)' &= ug \\ \int (uy)' &= \int ug \\ uy &= \int ug \end{aligned}$$

etc...

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.

Equilibrium Solution = constant solution

stable, unstable, semi-stable.

2.3, 2.5

Ch 1

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t,y)$, $y(t_0) = y_0$, solution may or may not exist and solution may or may not be unique.

But we have 2 theorems that guarantee both existence and uniqueness of solutions under certain conditions:

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

1st order differential equation (general case):

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

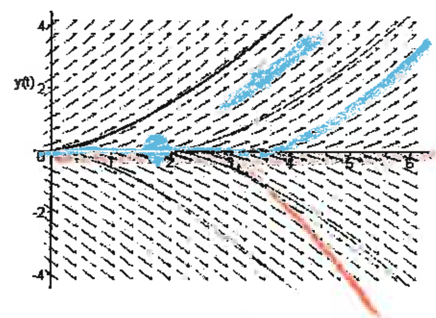
$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\frac{3}{2} y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.



Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

existance & uniquess

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

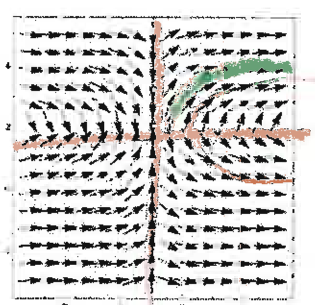
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-t)(2-y)))/sqrt(1 + 1/((1-t)(2-y))^2)$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int(2-y)dy = \int \frac{dt}{1-t}$ implies $2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t| + C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C > 0$ & $t \neq 1$ & $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

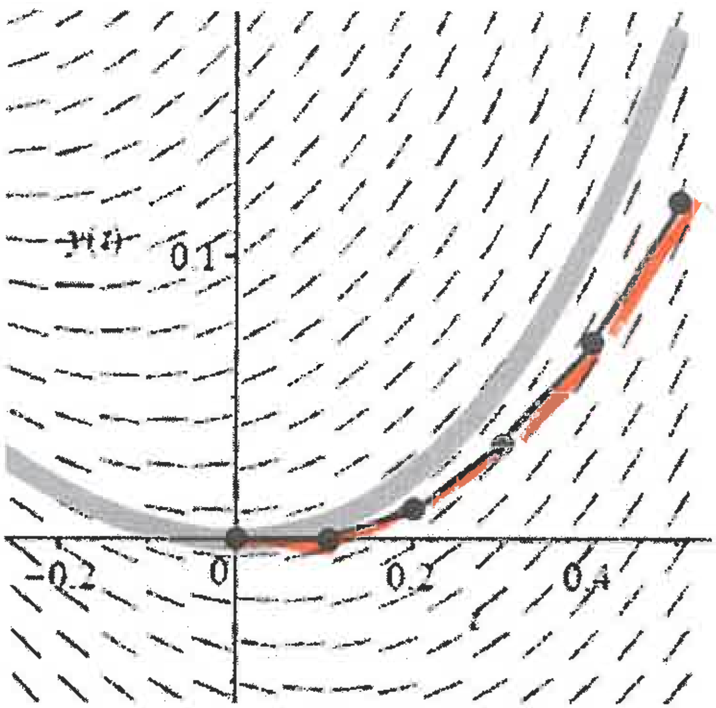
$1 - t < -e^{-\frac{C}{2}}$ or $1 - t > e^{-\frac{C}{2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

2.7: Approximating soln to IVP using multiple tangent lines.

Example: $y' = t + 2y$, $y(0) = 0$

$$y(t) = \begin{cases} 0 & 0 \le t \le 0.1 \\ 0.1t - 0.01 & 0.1 \le t \le 0.2 \\ 0.22t - 0.034 & 0.2 \le t \le 0.3 \\ 0.364t - 0.0772 & 0.3 \le t \le 0.4 \\ 0.5328t - 0.14672 & 0.4 \le t \le 0.5 \end{cases}$$



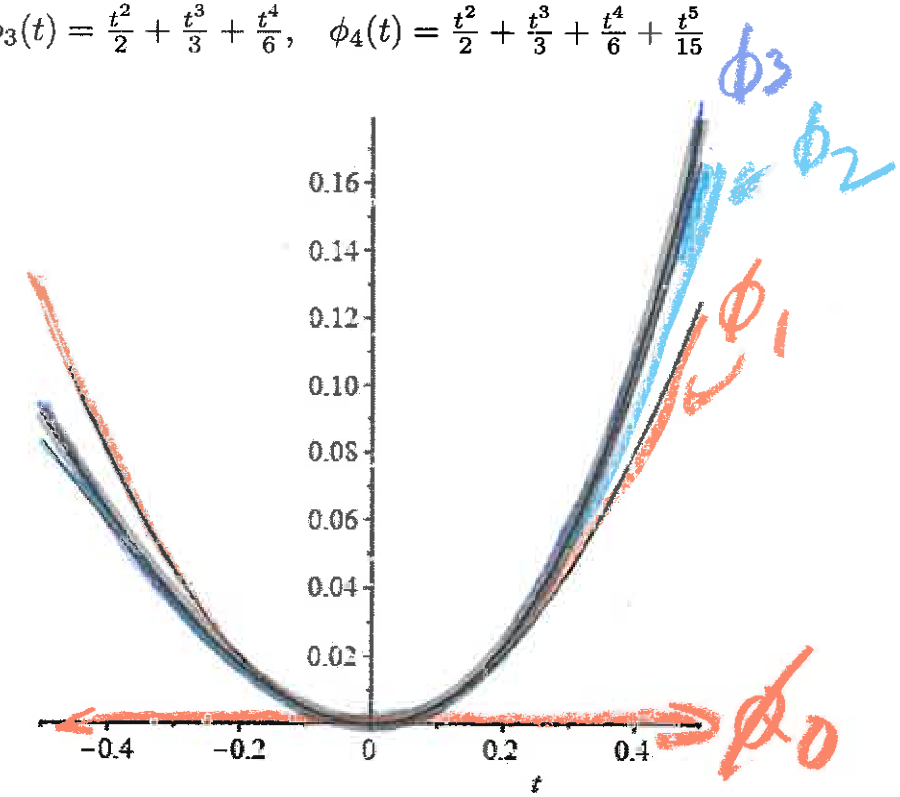
2.8: Approximating soln to IVP using seq of fns,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$$

Example: $y' = t + 2y$, $y(0) = 0$

$\phi_0(t) = 0$, $\phi_1(t) = \frac{t^2}{2}$, $\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$,

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$, $\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$

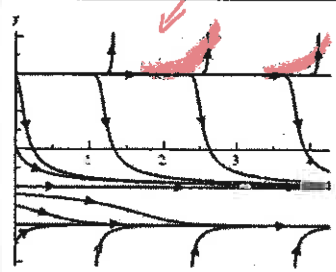


First Order Separable

- General Form: F(y) dy = G(x) dx
  - Usually looks like this in its starting form:
  - dy/dx = f(y) * f(x)
  - Can divide by f(y) and multiply both sides by dx to get above form
- Usually use when you need to divide or multiply by y to get it on the left
- Can directly integrate to solve
  - ~~∫ F(y) dy = ∫ G(x) dx~~
  - If y is in denominator, use natural log rule
  - Otherwise use normal integration

?

ugly



Equilibrium

- Stable
- Unstable
- Semi-stable
- Integral curves

Autonomous Equations

- General Form: dy/dt = f(y)
- Usually used to model populations or other logistic trends that taper off
- General Solution form: y = c * e^(rt) <-- Hopefully looks familiar
  - Note this is the solution in the special case of y' = ry
  - Note this special case is 1st order linear homogeneous y' - ry = 0.
- Can make direction graph to model the equation with different initial conditions

Direction Graphs

- Are often used with autonomous equations since there is only one variable to change
  - Can do with two variables but it takes longer
- When dy/dx = 0, you might have equilibrium solutions
  - Equilibrium solutions = constant solutions.
  - If y = c is a constant solution, then y' = 0.
  - y' = 0 implies f(t, y) = 0, but this may or may not give you constant solution(s).
- ~~Phase lines can't show the pattern of growth, but can show what direction the trends follow~~
- Phase lines can be used to determine long-term behavior of solutions.
- Phase lines can be misleading if IVP has no solution or multiple solutions.

Second Order

- General Form: y'' + p(t) y' + q(t) y = g(t)
  - Alternatively P(t) y'' + Q(t) y' + R(t) y = G(t)
  - But you can divide by P(t) to get above equation
  - But for 2nd and higher order, don't need to divide by P(t) before solving (unlike first order case when using integrating factor).
  - Homogeneous vs Non-Homogeneous
    - G(t) and g(t) equal 0 in formats above
  - Get into characteristic form:
    - ar^2 + br + c = 0
  - Solve for roots/put into e^rt format
  - Write as sum of two individual solutions to cover entire solution set

First Order Linear Differential Equation

$$ty' + (t+1)y = t \qquad y(\ln 2) = 1$$

$$y' + \frac{(t+1)}{t}y = 1$$

$$y' + \left(1+\frac{1}{t}\right)y = 1$$

→ integrating factor

$$e^{\int p(x)\,dx} = e^{\int (1+\frac{1}{t})dt} = e^{t+\ln t} = e^{t}\cdot \cancel{e}^{\ln t} = \underline{te^{t}}$$

$$\int te^{t}y' + te^{t}\left(1+\frac{1}{t}\right)y = \int te^{t} \qquad \begin{matrix} u = t \\ dv = e^{t}\end{matrix}$$

u	dv	~~B~~
t	e^t	$+$
1	e^t	$-$
0	e^t	$+$

$$te^{t}\cdot y = te^{t} - e^{t} + C$$

$$y = 1 - \frac{1}{t} + \frac{C}{te^{t}}$$

$$\cancel{1} = \cancel{1} - \frac{1}{\ln 2} + \frac{C}{2\ln 2}$$

$$\frac{1}{\ln 2} = \frac{C}{2\ln 2} \Rightarrow C = 2$$

$$\boxed{y(t) = 1 - \frac{1}{t} + \frac{2}{te^{t}}}$$

$$t \neq 0 \rightarrow t \in (-\infty, 0) \cup (0, \infty)$$

↓

$t \in (0, \infty)$ since $0 < \ln 2 < \infty$

Separable Equation

$2y-5=0 \Rightarrow y \neq \frac{5}{2}$

2

$$\frac{dy}{dx} = \frac{3x^2-e^x}{2y-5} \qquad y(0)=1$$

$$\int (2y-5)\,dy = \int (3x^2-e^x)\cdot dx$$

$$y^2-5y = x^3-e^x+C$$

$$(1)^2-5(1) = \cancel{0}^{0}-\cancel{e^0}^{1}+C \Rightarrow C=-3$$

$$y^2-5y = x^3-e^x-3 \quad \longleftarrow \text{implicit}$$

$$y^2-5y+b^2 = x^3-e^x-3+b^2 \qquad b=\frac{-5}{2}$$

$$y^2-5y+\frac{25}{4} = x^3-e^x-3+\frac{25}{4}$$

$$\left(y-\frac{5}{2}\right)^2 = x^3-e^x+\frac{13}{4}$$

$$y-\frac{5}{2} = \pm\sqrt{x^3-e^x+\frac{13}{4}}$$

$$\boxed{y = \frac{5}{2}+\sqrt{x^3-e^x+\frac{13}{4}}}$$

choose $+$ since

$$y(0) = \frac{5}{2}+\sqrt{0-e^0+\frac{13}{4}} = \frac{5}{2}+\sqrt{\frac{9}{4}} = 1$$

while

$$\frac{5}{2}-\sqrt{0-e^0+\frac{13}{4}} \neq 1$$

$$x^3-e^x+\frac{13}{4} \geq 0$$

and $y \neq \frac{5}{2}$

$$\Rightarrow x^3-e^x+\frac{13}{4} > 0$$

Note the convention in this class is to exclude points from domain where tangent line is vertical (so no dividing by 0)

we also want our sol'n to be a function (thus choose +/- depending on initial value)

3

At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of 3 gal/min, and that the well-stirred solution is leaving the tank at the same rate. Find an expression for the amount of salt, $Q(t)$, in the tank at any time t.

$$* \quad Q(0) = Q_0 \qquad \text{In} = \frac{1}{4}\frac{\text{lb}}{\text{gal}} * 3\frac{\text{gal}}{\text{min}}$$

$$100 \text{ gal} = V \qquad \text{Out} = 3\frac{\text{gal}}{\text{min}} * \frac{Q}{100}\frac{\text{lb}}{\text{gal}}$$

$$\frac{dQ}{dt} = \frac{1}{4}(3) - 3\left(\frac{Q}{100}\right)$$

$$Q' + \frac{3Q}{100} = \frac{3}{4}$$

$$u(t)\,Q' + u(t)\frac{3Q}{100} = u(t)\frac{3}{4}$$

$$u(t) = e^{\int \frac{3}{100}\,dt} = e^{(3/100)t}$$

$$e^{3/100\,t}\,Q' + e^{3/100\,t} * \frac{3Q}{100} = e^{\frac{3}{100}t} * \frac{3}{4}$$

$$\int\left(e^{\frac{3}{100}t} * y\right)' = \int e^{\frac{3}{100}t} * \frac{3}{4}$$

continues on ⟹ 3.5

$$\left(e^{\frac{3}{100}t} * y\right)'$$

$$\underbrace{\frac{3y}{100} e^{\frac{3}{100}t} + y' \cdot e^{3/100\,t}}$$

Product Rule checks ✓ ü

continues on ⟹ 3.6

3.5

$$e^{\frac{3}{100}t} * y = \frac{100}{\cancel{3}} e^{\frac{3}{100}t} * \frac{\cancel{3}}{4} = 25 e^{3/100\, t} + C$$

$$Q = 25 + C / e^{3/100\, t}$$

$$Q(0) = Q_0$$

$$Q_0 = 25 + C$$

$$Q_0 - 25 = C$$

$$\boxed{Q = 25 + (Q_0 - 25) / e^{3/100 * t}}$$

$$Q = 25 + (Q_0 - 25) e^{\frac{-3t}{100}}$$

3.6

OR

$$2y'' - 3y' + y = 0 \Rightarrow 2r^2 - 3r + 1 = 0$$

4

$$y'' - \frac{3}{2}y' + \frac{y}{2} = 0$$

$$\Rightarrow r = \frac{3 \pm \sqrt{9 - 4(2)(1)}}{2(2)}$$

$$r^2 - \frac{3}{2}r + \frac{1}{2} = 0$$

OR

$$r = \frac{\frac{3}{2} \pm \sqrt{(-3/2)^2 - 4(1)(\frac{1}{2})}}{2(1)}$$

$$2r^2 - 3r + 1 = (2r-1)(r-1)$$

$$r = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}}{2}$$

$$r = 1, \ 1/2$$

$$y_1 = e^{(1)t} \qquad y_2 = e^{(1/2)t}$$

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 e^t + c_2 e^{t/2}}$$

$y(0)=0 \quad y'(0)=0$

5

(1) $0 = c_1 + c_2 \qquad e^0 = 1$

$y' = c_1 e^t + \frac{1}{2} c_2 e^{t/2}$

(2) $0 = c_1 + \frac{1}{2} c_2$

$c_1 = -c_2$

$c_1 = -\frac{1}{2} c_2 \qquad c_2 = 0 \Rightarrow c_1 = 0$

$y = \frac{5}{2} + \sqrt{x^3 - e^x + \frac{13}{4}} \qquad y(0) = 1$

$1 = \frac{5}{2} \pm \sqrt{-1 + \frac{13}{4}}$

$1 - \frac{5}{2} = \frac{-3}{2} = \pm \sqrt{\frac{9}{4}} = -\frac{3}{2}$

OK, but related to earlier problem

6

Solve the equation

$$\frac{dy}{dt} = y(2-y)$$

and ~~draw~~ find the equilibrium solutions and classify them as asymptotically stable or unstable.

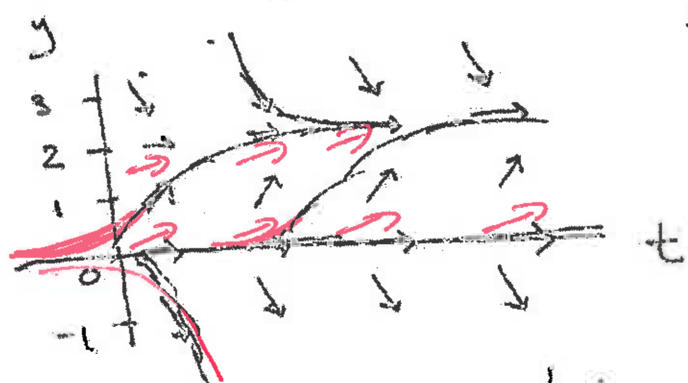
$0 = y(2-y)$:

$y = 0$ or 2

$\frac{dy}{dt} = 1(2-1) = 1$

$\frac{dy}{dt} = -1(2+1) = -3$

$\frac{dy}{dt} = 3(-1) = -3$



$y = 0$ unstable

$y = 2$ stable

$$\frac{dy}{dt} = y(2-y)$$

$$\int \frac{dy}{y(2-y)} = \int dt$$

$$\frac{A}{y} + \frac{B}{2-y} = \frac{1}{y(2-y)}$$

$$\frac{(2-y)A + By}{y(2-y)} = \frac{1}{y(2-y)}$$

$$(2-y)A + By = 1$$

$$A = \frac{1}{2} \qquad B = \frac{1}{2}$$

$$(2-y)\frac{1}{2} + \frac{1}{2}y = 1 - \cancel{\frac{y}{2} + \frac{y}{2}} = 1$$

$$\int \left(\frac{1/2}{y} + \frac{1/2}{2-y} \right) dy = \int dt$$

$$\frac{1}{2}\ln y - \frac{1}{2}\ln(2-y) = t + C$$

$$\frac{1}{2}\ln\left(\frac{y}{2-y}\right) = t + C \qquad C = 2C_1$$

$$e^{\ln\left(\frac{y}{2-y}\right)} = e^{2t + C_1} \qquad C_2 = e^{C_1}$$

$$\boxed{\frac{y}{2-y} = e^{2t} \cdot C_2}$$

$$-1 \cdot \frac{y}{y-2}$$

Solve for y

$C = \underline{+}$

Derivation of general solutions:

Ch 3: we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.1: If $b^2 - 4ac > 0$: $\quad y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$

$= (c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$

$\quad = k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$

3.4 repeated root

2nd order LINEAR HOMOG

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $\boxed{y = v(t) e^{rt}?}$ Guess similar 3.6

unknown fn — homog. soln

This is a simple case of variation of params see §3.6

$y' = v'(t) e^{rt} + v(t) r e^{rt}$

$y'' = v''(t) e^{rt} + v'(t) r e^{rt} + v'(t) r e^{rt} + v(t) r^2 e^{rt}$

$\quad = v''(t) e^{rt} + 2v'(t) r e^{rt} + v(t) r^2 e^{rt}$

$ay'' + by' + cy = 0$

$a(v'' e^{rt} + 2v' r e^{rt} + v r^2 e^{rt}) + b(v' e^{rt} + v r e^{rt}) + cv e^{rt} = 0$

$a(v''(t) + 2v'(t) r + v(t) r^2) + b(v'(t) + v(t) r) + cv(t) = 0$

$av''(t) + 2av'(t) r + av(t) r^2 + bv'(t) + bv(t) r + cv(t) = 0$

$av''(t) + (2ar + b) v'(t) + (ar^2 + br + c) v(t) = 0$

$= 0 \leftarrow e^{rt}$ soln to homogen

$av''(t) + (2a(\frac{-b}{2a}) + b) v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (\cancel{-b + b}) v'(t) = 0. \qquad$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t) e^{r_1 t} = (k_1 t + k_2) e^{r_1 t}$ is a soln

repeated root

$r = \frac{-b \pm \sqrt{0}}{2a}$

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

for 3.4 repeated root case