Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$. ← imaginary

RECOMMENDED Method:

Since $r = 0 \pm 1i$, $y = k_1cos(1t) + k_2sin(1t)$ ← general sol'n

Then $y' = -k_1sin(t) + k_2cos(t)$

IVP

$y(0) = -1$: $-1 = k_1cos(0) + k_2sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1sin(0) + k_2cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

NOT RECOMMENDED: work with $y = c_1e^{it} + c_2e^{-it}$ ← general sol'n not simplified

IVP

$y' = ic_1e^{it} - ic_2e^{-it}$

$y(0) = -1$: $-1 = c_1e^0 + c_2e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $-3 = ic_1e^0 - ic_2e^0$ implies $-3 = ic_1 - ic_2$.

$-1i = ic_1 + ic_2$.

$-3 = ic_1 - ic_2$.

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i-i^2}{-2} = \frac{3i-1}{2}$

$2ic_2 = 3 - i$ implies $c_2 = \frac{3i-i^2}{-2} = \frac{-3i-1}{2}$

If $y(0) = -1$, $y'(0) = -3$, Sol'n is real-valued

Euler's formula: $e^{ix} = cos(x) + isin(x)$

IVP sol

$y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it} = (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$

$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$

$= (\frac{3i}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{-1}{2})isin(t) + (\frac{-3i}{2})cos(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - (\frac{-1}{2})isin(t)$

$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$

$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$

y $= -3sin(t) - 1cos(t)$ ← simplified sol'n

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$.
There is a unique choice of constants c_1 and c_2 such that $c_1\phi_1 + c_2\phi_2$ satisfies this homog linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_0'$.

iff

$W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0.$

Thm 3.2.4: Given the hypothesis of thm 3.2.1, suppose that ϕ_1 and ϕ_2 are two solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Defn If ϕ_1 and ϕ_2 satisfy the conditions in thm 3.2.4, then ϕ_1 and ϕ_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

FYI: Linear Independence and the Wronskian

Defn: ϕ_1 and ϕ_2 are linearly dependent if there exists constants c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$ and

$$c_1\phi_1(t) + c_2\phi_2(t) = 0 \text{ for all } t \in (a,b)$$

Thm ~~3.3.1~~: If $\phi_1 : (a,b) \to R$ and $\phi_2(a,b) \to R$ are differentiable functions on (a, b) and
if $W(\phi_1, \phi_2)(t_0) \neq 0$ for some $t_0 \in (a,b)$, then
ϕ_1 and ϕ_2 are linearly independent on (a,b).
Moreover, if ϕ_1 and ϕ_2 are linearly dependent on (a,b), then $W(\phi_1, \phi_2)(t) = 0$ for all $t \in (a,b)$

Proof idea:

If $c_1\phi_1(t) + c_2\phi_2(t) = 0$ for all $t \in (a,b)$,
then $c_1\phi_1'(t) + c_2\phi_2'(t) = 0$ for all $t \in (a,b)$

deriv

Solve the following linear system of equations for c_1, c_2

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) = 0$$
$$c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = 0$$

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

unique soln $\Leftrightarrow$ ϕ_1, ϕ_2 indep

In other words the fundamental set of solutions $\{\phi_1, \phi_2\}$ to $y'' + p(t)y' + q(t)y = 0$ form a basis for the set of all solutions to this linear homogeneous DE.

$$\Leftrightarrow \begin{vmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{vmatrix} \neq 0$$

$\Leftrightarrow$ IVP has unique soln

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$.

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to
$ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $\quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $\quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Wronskian: determinant of coef matrix used to find c_i's in IVP at arb t_0

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0), y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{matrix} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{matrix} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) $W(cos(t), sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) $W(e^{dt}cos(nt), e^{dt}sin(nt)) =$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(nt$

$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$

$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$

$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$

$= ne^{2dt}[cos^2(nt) + sin^2(nt)] = ne^{2dt} > 0$ for all t.

Gen Soln: $y = c_1\phi_1 + c_2\phi_2$ for 2nd order linear homo

Existence and Uniqueness for LINEAR DEs.

<u>Homogeneous:</u>

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = 0$$

<u>Non-homogeneous:</u> $g(t) \neq 0$

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = g(t)$$

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the

IVP: $y' + p(t)y = g(t), \quad y(t_0) = y_0$

Thm: If $y = \phi_1(t)$ is a solution to <u>homogeneous</u> equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to <u>non-homogeneous</u> equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

Partial proof: $y = \phi_1(t)$ is a solution to $y' + p(t)y = 0$ implies

Thus $y = c\phi_1(t)$ is a solution to $y' + p(t)y = 0$ since

$y = \psi(t)$ is a solution to $y' + p(t)y = g(t)$ implies

Thus $y = c\phi_1(t) + \psi(t)$ is a solution to $y' + p(t)y = g(t)$ since

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$\begin{aligned} y'' + p(t)y' + q(t)y &= g(t), \\ y(t_0) &= y_0, \\ y'(t_0) &= y'_0 \end{aligned}$$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a <u>homogeneous</u> linear differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where p and q are functions of t (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

Note: You must be able to identify which techniques you need to use. For example:

Integration:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.

For differential equations:

Is the differential equation 1rst order or 2nd order?

If 2nd order: Section 3.1, solve $ay'' + by' + cy = 0$.

Guess $y = e^{rt}$.

characteristic eqn

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$.

Need to have two independent solutions.

If $y = \phi_1, y = \phi_2$ are solutions to a LINEAR HOMOGENEC differential equation, $y = c_1\phi_1 + c_2\phi_2$ is also a solution

If 1st order: Is the equation linear or separable or ? or Bernoulli

IVP ⇐ plug in to find constants

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

Check

$$\begin{aligned} y' + py &= g \\ y'u + upy &= ug \\ (uy)' &= ug \\ \int (uy)' &= \int ug \\ uy &= \int ug \end{aligned}$$

etc...

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.

Equilibrium Solution = constant solution

stable, unstable, semi-stable.

2.3, 2.5

Ch

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t,y)$, $y(t_0) = y_0$, solution may or may not exist and solution may or may not be unique.

But we have 2 theorems that guarantee both existence and uniqueness of solutions under certain conditions:

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

1st order differential equation (general case):

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

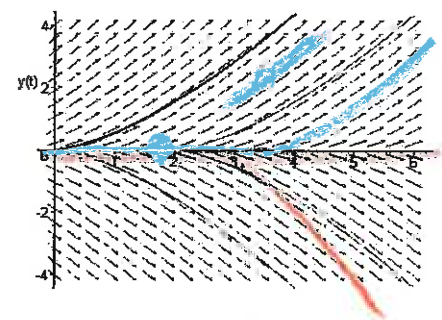
$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}}dy = dt$$
$$\tfrac{3}{2}y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.



Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3}y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

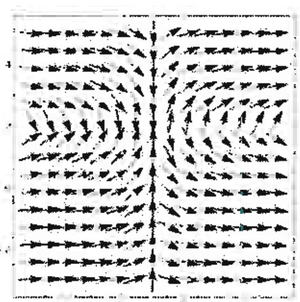
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-t)(2-y)))/sqrt(1 + 1/((1-t)(2-y))^2)$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int (2-y)dy = \int \frac{dt}{1-t}$ implies $2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t| + C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$ & $t \neq 1$ & $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

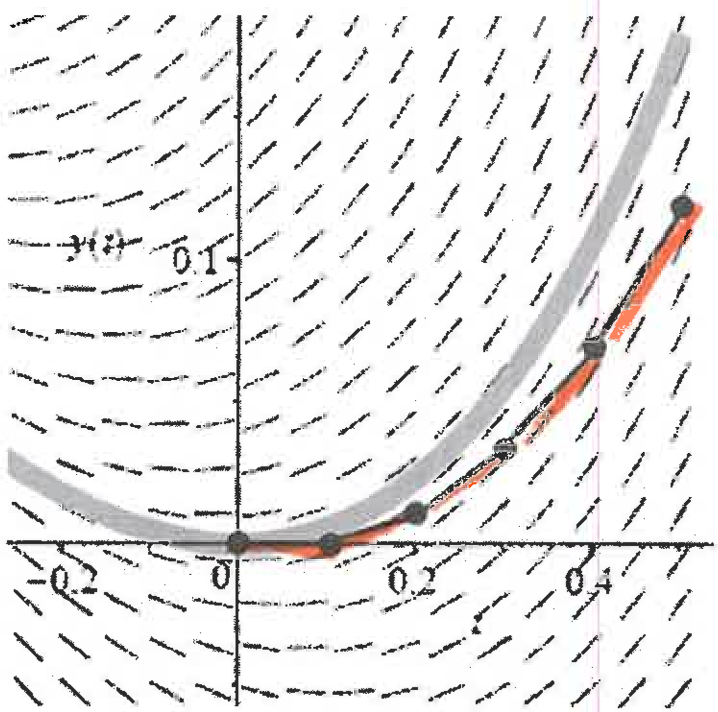
$1 - t < -e^{-\frac{C}{2}}$ or $1 - t > e^{-\frac{C}{2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

2.7: Approximating soln to IVP using multiple tangent lines.

Example: $y' = t + 2y$, $y(0) = 0$

$$y(t) = \begin{cases} 0 & 0 \le t \le 0.1 \\ 0.1t - 0.01 & 0.1 \le t \le 0.2 \\ 0.22t - 0.034 & 0.2 \le t \le 0.3 \\ 0.364t - 0.0772 & 0.3 \le t \le 0.4 \\ 0.5328t - 0.14672 & 0.4 \le t \le 0.5 \end{cases}$$



2.8: Approximating soln to IVP using seq of fns,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$$

Example: $y' = t + 2y$, $y(0) = 0$

$\phi_0(t) = 0, \quad \phi_1(t) = \frac{t^2}{2}, \quad \phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3},$

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}, \quad \phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$