MATRIX multiplication is LINEAR $\begin{cases} A(x+y) = Ax + Ay \\ A(cx) = cAx \end{cases}$

Linear Functions

A function f is linear if $\boxed{f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})}$

Or equivalently f is linear if 1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0 + 1) = f(1) = 5$, but $f(0) + f(1) = 3 + 5 = 8$. Hence $f(0+1) \neq f(0) + f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \end{bmatrix}$$

Question: When is a line, $f(x) = mx + b$, a linear function?

Example 2.) $f : R^2 \to R^2$,
$f((x_1, x_2)) = (2x_1, x_1 + x_2)$ ← MATRIX ALGEBRA

FYI

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x} + b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) =$
$(ax_1 + by_1, ax_2 + by_2)$

$f(ax_1 + by_1, ax_2 + by_2)$

$= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)$

$= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)$

$= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)$

$= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)$

$= af((x_1, x_2)) + bf((y_1, y_2))$

Example 3.) D : set of all differential functions → set of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

In this class

scalar = constant real #

Example 4.) Given a, b real numbers, I : set of all integrable functions on [a, b] $\to R$, $I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

will use in Ch 6

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and $f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Ch 3

Example 6.) D : set of ~~all twice differential functions~~ $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$
$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$
$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$
$$= sL(f) + tL(g)$$

$ay'' + by' + cy = 0$

Ch 3.1, 2, 4

$ay'' + by' + cy = 0$

$L(\psi_i) = a\psi_i'' + b\psi_i' + c\psi_i = 0$

LINEAR COMB OF SOL NS ARE SOL NS TO LINEAR HOMOG DE

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

$$\text{Hence } L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$$
$$= 3(0) + 5(0) = 0.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

$$\text{Hence } L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$$
$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$$af'' + bf' + cf = 3h + 5k$$

non homog

3.5, 3.6

Summary of sections 3.1, 3, 4: Solve linear homogeneous 2nd order DE with constant coefficients.

a, b, c constants

Solve $ay'' + by' + cy = 0$ Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $\boxed{ar^2 + br + c = 0}$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

3.1 2 real soln's

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

3.3 2 complex solns $r = d \pm in$

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula. $r = d \pm in$

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

3.4 1 real repeated root

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

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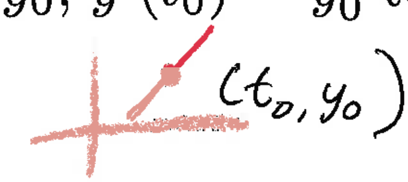
slope at (t_0, y_0)

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for c_1, c_2 to find unique solution.



Examples:

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

two real solns
3, 1

$r^2e^{rt} - 3re^{rt} - 4e^{rt} = 0$

$r^2 - 3r - 4 = 0$ implies $(r-4)(r+1) = 0$. Thus $r = 4, -1$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP:
Need to solve for 2 unknowns, c_1 & c_2; thus need 2 eqns:

$y = c_1e^{4t} + c_2e^{-t}$, $\quad y(0) = 1 \quad$ implies $\quad 1 = c_1 + c_2$

$y' = 4c_1e^{4t} - c_2e^{-t}$, $\quad y'(0) = 2 \quad$ implies $\quad 2 = 4c_1 - c_2$

Thus $3 = 5c_1$ & hence $c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

Thus IVP soln: $y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$

Ex 2: Solve $y'' - 3y' + 4y = 0$. Sect 3.3

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

2 complex solns

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1e^{\frac{3}{2}t}cos(\frac{\sqrt{7}}{2}t) + c_2e^{\frac{3}{2}t}sin(\frac{\sqrt{7}}{2}t)$

Ex 3: $y'' - 6y' + 9y = 0$ implies $r^2 - 6r + 9 = (r-3)^2 = 0$
Repeated root, $r = 3$ implies

sect 3.4

general solution is $y = c_1e^{3t} + c_2te^{3t}$

So why did we guess $y = e^{rt}$?

FYI

Goal: Solve linear homogeneous 2nd order DE with constant coefficients,

$$ay'' + by' + cy = 0 \text{ where } a, b, c \text{ are constants}$$

Standard mathematical technique: make up simpler problems and see if you can generalize to the problem of interest.

Ex: linear homogeneous 1rst order DE: $y' + 2y = 0$

integrating factor $u(t) = e^{\int 2dt} = e^{2t}$

$y'e^{2t} + 2e^{2t}y = 0$

$(e^{2t}y)' = 0$. Thus $\int (e^{2t}y)'dt = \int 0dt$. Hence $e^{2t}y = C$

So $y = Ce^{-2t}$.

Thus exponential function could also be a solution to a linear homogeneous 2nd order DE

Ex: Simple linear homog 2nd order DE $y'' + 2y' = 0$.

Let $v = y'$, then $v' = y''$

$y'' + 2y' = 0$ implies $v' + 2v = 0$ implies $v = e^{2t}$.

Thus $v = y' = \frac{dy}{dt} = Ce^{-2t}$. Hence $dy = Ce^{-2t}dt$ and

$$y = c_1e^{-2t} + c_2$$

$$y = c_1 e^{-2t} + c_2.$$

$y'' + 2y' = 0$
$r^2 + 2r = 0$
$r(r+2) = 0$

Note 2 integrations give us 2 constants.

Note also that we the general solution is a linear combination of two solutions:

$r = -2$

Let $c_1 = 1,\ c_2 = 0$, then we see, $y(t) = e^{-2t}$ is a solution.

Let $c_1 = 0,\ c_2 = 1$, then we see, $y(t) = 1$ is a solution.

$y(t) = e^{0t}$ $r = 0$

The general solution is a linear combination of two solutions:

$$y = c_1 e^{-2t} + c_2(1).$$

Recall: you have seen this before:

Solve linear homogeneous matrix equation $A\mathbf{y} = \mathbf{0}$.

The general solution is a linear combination of linearly independent vectors that span the solution space:

$$\mathbf{y} = c_1\mathbf{v_1} + \ldots c_n\mathbf{v_n}$$

FYI: You could see this again:

Math 4050: Solve homogeneous linear recurrance relation $x_n - x_{n-1} - x_{n-2} = 0$ where $x_1 = 1$ and $x_2 = 1$.

Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

Note $x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: $x_n = x_{n-1} + x_{n-2}$ implies $x_n - x_{n-1} - x_{n-2} = 0$

Guess

Suppose $x_n = r^n$. Then $x_{n-1} = r^{n-1}$ and $x_{n-2} = r^{n-2}$

Then $0 = x_n - x_{n-1} - x_{n-2} = r^n - r^{n-1} - r^{n-2}$

Thus $r^{n-2}(r^2 - r - 1) = 0$.

Thus either $r = 0$ or $r = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1\pm\sqrt{5}}{2}$

Thus $x_n = 0$, $\quad x_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homog linear recurrence relation: $x_n - x_{n-1} - x_{n-2} = 0$.

LINEAR COMB

Hence $x_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies this

homogeneous linear recurrence relation.

Suppose the initial conditions are $x_1 = 1$ and $x_2 = 1$

Then for $n = 1$: $x_1 = 1$ implies $c_1 + c_2 = 1$

For $n = 2$: $x_2 = 1$ implies $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

We can solve this for c_1 and c_2 to determine that

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$