Proof outline of thm 2.4.2: existance & uniqueness

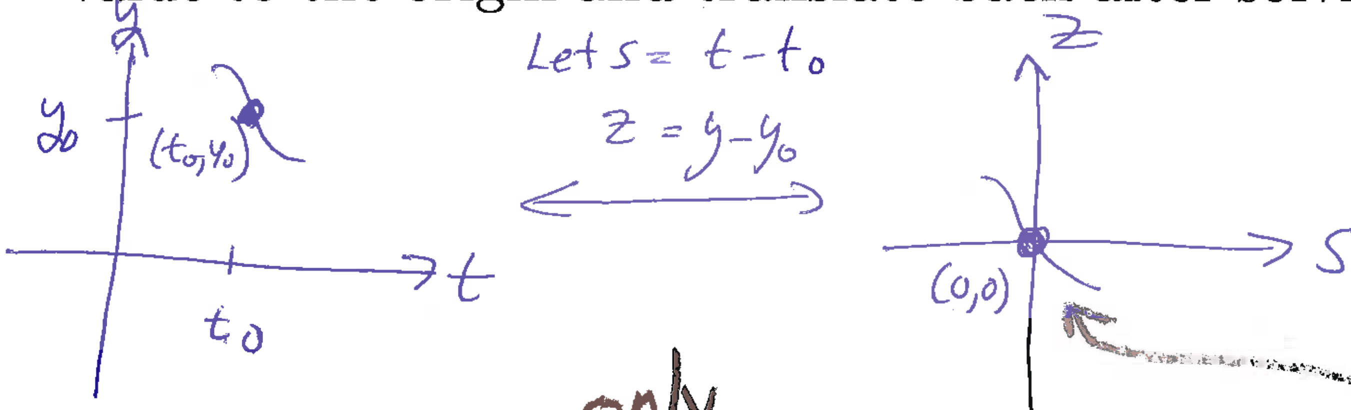
2.8: Approximating solution using

Method of Successive Approximation

(also called Picard's iteration method).

IVP: $y' = f(t, y)$, $y(t_0) = y_0$.

Note: Can always translate IVP to move initial value to the origin and translate back after solving:



Hence for simplicity only **in section 2.8**, we will assume initial value is at the origin: $y' = f(t, y)$, $y(0) = 0$.

Thm 2.4.2: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Thm 2.8.1 is translated to origin version of Thm 2.4.2:

Thm 2.8.1: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous for all t in $(-a, a) \times (-c, c)$,

then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(0) = 0.$$

Proof outline (note this is a constructive proof and thus the proof is very useful).

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)

f, $\partial f / \partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$.

Then $y = \phi(t)$ is a solution to (*) iff

plug it in

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

integrate both sides

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s))ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))ds$ simplify FTC

Thus $y = \phi(t)$ is a solution to (*)

iff $\phi(t) = \int_0^t f(s, \phi(s))ds$

Useless formula but can use to find sequence of functions $\rightarrow \phi$

$\phi(t) = \int_0^t f(s, \phi(s))\, ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\boxed{\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds}$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ ← the sol'n to $y' = f(t,y)$

To finish the proof, need to answer the following questions (see book or more advanced class):

1.) Does $\phi_n(t)$ exist for all n? ← ✓ yes since f cont for all t, y near $(0,0)$

2.) Does sequence ϕ_n converge? ← See more advanced class for general case

FYI: RATIO TEST ← Don't need for this class

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

FYI: For specific cases, can plug in

4.) Is the solution unique.

see book

$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$

Example: $y' = t + 2y$. That is $\boxed{f(t,y) = t + 2y = y'}$

Let $\phi_0(t) = 0 \Leftarrow$ constant 0 fn

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$\phi_1(t) = \frac{t^2}{2}$

$= \int_0^t s ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$

$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

Let $\phi_4(t) = \int_0^t f(s, \phi_3(s))ds$

$= \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6})ds$

$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6}))ds$

$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$

$\vdots$

Determine formula for ϕ_n:

LOOK FOR FACTORIALS

Note patterns:

$\int_0^t s\,ds = \frac{t^2}{2} = \frac{t^2}{2!}$

$\int_0^t \frac{s^2}{2} ds = \frac{t^3}{3\cdot 2} = \frac{t^3}{3!}$

$\int_0^t \frac{s^3}{3\cdot 2} ds = \frac{t^4}{4\cdot 3\cdot 2} = \frac{t^4}{4!}$

$\int_0^t \frac{s^4}{4\cdot 3\cdot 2} ds = \frac{t^5}{5\cdot 4\cdot 3\cdot 2} = \frac{t^5}{5!}$

Thus look for factorials.

$\text{Soln} = \lim_{n\to\infty} \phi_n$

$\phi(t) = \sum_{K=2}^{\infty} \frac{2^{K-2} t^K}{K!}$

$\phi_0(t) = 0$

$\phi_1(t) = \frac{t^2}{2} = \sum_{K=2}^{2} \frac{2^{K-2} t^K}{K!} = \frac{2^0 t^2}{2!}$ ✓

$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} = \sum_{K=2}^{4} \frac{2^{K-2} t^K}{K!} = \frac{2^0 t^2}{2!} + \frac{2 t^3}{3!} + \frac{2^2 t^4}{4!}$ ✓

$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{3\cdot 2} + \frac{t^5}{5\cdot 3}$

Thus $\phi_n(t) = \sum_{K=2}^{n+1} \frac{2^{K-2}}{K!} t^K$

FYI (ie not on quizzes/exam):

Defn: $\displaystyle\sum_{k=0}^{\infty} a_k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

Taylor's Theorem: If f is analytic at 0, then for small x (i.e., x near 0),

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots$$

Example:

$e^t = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!}$ and thus $e^{bt} = \displaystyle\sum_{k=0}^{\infty} \frac{b^k t^k}{k!}$ for t near 0.

$$\phi_n(t) = \sum_{k=2}^{n} \frac{2^{k-2}}{k!} t^k$$

Thus $\phi(t) = \displaystyle\lim_{n\to\infty} \phi_n(t) = \frac{1}{4}\sum_{k=2}^{\infty} \frac{4 \cdot 2^{k-2}}{k!} t^k = \frac{1}{4}\sum_{k=2}^{\infty} \frac{2^k}{k!} t^k$

$$\phi(t) = \frac{1}{4}\left(e^{2t} - \underset{k=0}{1} - \underset{k=1}{2t}\right)$$

Alternatively solve 6 IVP to find ϕ