Integration by parts:

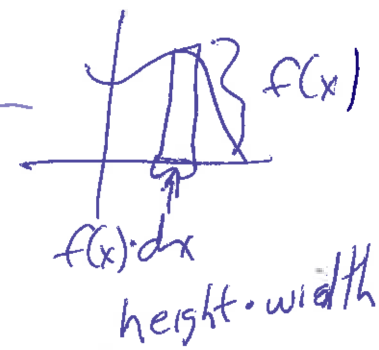
Derivative of a product: $\boxed{(uv)' = uv' + vu'}$ product rule

$$uv' = (uv)' - vu'$$

$$\int uv' = \int (uv)' - \int vu'$$

$$\boxed{\int uv' = (uv) - \int vu'}$$

Example: $\int e^{2x} sin(3x)\,dx$



Let $u = sin(3x)$, $dv = e^{2x}$

then $du = 3cos(3x)$, $v = \frac{1}{2}e^{2x}$

then $d^2u = -9sin(3x)$, $\int v = \frac{1}{4}e^{2x}$

$$\int e^{2x} sin(3x)\,dx = \frac{1}{2}sin(3x)e^{2x} - \int \frac{3}{2}e^{2x}cos(3x)\,dx$$

$$= \frac{1}{2}sin(3x)e^{2x} - [\frac{3}{4}cos(3x)e^{2x} - \int \frac{-9}{4}sin(3x)e^{2x}]\,dx$$

$$\int e^{2x} sin(3x)\,dx = \frac{1}{2}sin(3x)e^{2x} - \frac{3}{4}cos(3x)e^{2x} - \frac{9}{4}\int sin(3x)e^{2x}\,dx$$

$$\frac{13}{4}\int e^{2x} sin(3x)\,dx = \frac{1}{2}sin(3x)e^{2x} - \frac{3}{4}cos(3x)e^{2x}$$

$$\int e^{2x} sin(3x)\,dx = \frac{4}{13}[\frac{1}{2}sin(3x)e^{2x} - \frac{3}{4}cos(3x)e^{2x}]$$

Optional Exercise: Calculate $\int e^{x} cos(2x)$

unique way to represent elements in span of $b_1, \dots, b_n$

Linear algebra pre-requisites you must know.

$\mathbf{b_1}, \dots, \mathbf{b_n}$ are linearly independent if

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \dots + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + \dots + d_n\mathbf{b_n}$$
$$\text{implies } c_1 = d_1,\ c_2 = d_2 \dots,\ c_n = d_n.$$

or equivalently,

$\mathbf{b_1}, \dots, \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \dots + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = \dots c_n$.

Example 1: $\mathbf{b_1} = (1, 0, 0)$, $\mathbf{b_2} = (0, 1, 0)$, $\mathbf{b_3} = (0, 0, 1)$.

$$(1, 2, 3) \neq (1, 2, 4).$$

$$\text{If } (a, b, c) = (1, 2, 3) \text{ then } a = 1,\ b = 2,\ c = 3.$$

Example 2: $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$ then $a = 1,\ b = 2,\ c = 3$.

unique way to represent elements

$\frac{Dx^2+Ex+F}{x^3}$

1 degree less than bottom

Application: Partial Fractions

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$

If you don't like denominators, get rid of them:

$x=3$: $4 = 0 + C(10) \Rightarrow C = \frac{4}{10} = \frac{2}{5}$

$$4 = (Ax+B)(x-3) + C(x^2+1)$$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A+C)x^2 + (B-3A)x - 3B + C$$

I.e., $0x^2+0x+4 = (A+C)x^2 + (B-3A)x - 3B + C$

Thus $\boxed{0 = A+C,}$ $\boxed{0 = B-3A,}$ $\boxed{4 = -3B+C.}$

$C = -A$, $B = 3A$, $4 = -3(3A) + -A \Rightarrow 4 = -10A$.

Hence $A = -\frac{2}{5}$, $B = 3(-\frac{2}{5}) = -\frac{6}{5}$, $C = \frac{2}{5}$.

$$\text{Thus, } \int \frac{4}{(x^2+1)(x-3)}\,dx = \int \frac{-\frac{2}{5}x - \frac{6}{5}}{x^2+1} + \int \frac{(\frac{2}{5})\cdot 5}{(x-3)\cdot 5}$$

$$= \int \frac{-2x-6}{5(x^2+1)}\,dx + \int \frac{2}{5(x-3)}\,dx$$

Alternatively, can plug in $x = 3$ to quickly find C and then solve for A and B. Can also use matrices to solve linear eqns.