Math 3600 Differential Equations Exam #1 2
March 2, 2016

SHOW ALL WORK

[12] 1.) Find the largest eigenvalue and its corresponding eigenvectors for $\begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix}$

$\begin{vmatrix} 5-r & 2 \\ 3 & 0-r \end{vmatrix} = (5-r)(-r) - 6 = r^2 - 5r - 6 = (r+1)(r-6)$. Thus $r = -1, 6$

For $r = 6$: $\begin{bmatrix} 5-r & 2 \\ 3 & 0-r \end{bmatrix} = \begin{bmatrix} 5-6 & 2 \\ 3 & 0-6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Answer: The largest eigenvalue of the above matrix is _6_

and its eigenvectors are all non-zero multiples of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

[8] 2.) Find all the singular points of the following differential equation and determine whether each one is regular or irregular.

$$x^3(x-3)y'' - 6xy' + 9xy = 0$$

x = 0 can't solve

x^2

$1y'' - \frac{6}{x^2(x-3)}y' + \frac{9}{x^2(x-3)}y = 0$. Thus $x = 0, 3$ are singular points.

Euler equation: $x^2y'' + \alpha xy' + \beta y = 0$.

Multiply by x^2: $x^2y'' - \left(\frac{6}{x(x-3)}\right)xy' + \left(\frac{9}{(x-3)}\right)y = 0$. Thus $x = 0$ is an irregular singular point.

Multiply by $(x-3)^2$: $(x-3)^2y'' - \left(\frac{6}{x^2}\right)(x-3)y' + \left(\frac{9(x-3)}{x^2}\right)y = 0$. Thus $x = 3$ is an regular singular point.

Alternately: $lim_{x\to 0}\, x\left(\frac{6}{x^2(x-3)}\right)$ is not finite. Thus $x = 0$ is an irregular singular point.

x^2 $\frac{9}{x^2(x-3)}$

$lim_{x\to 3}\, (x-3)\left(\frac{6}{x^2(x-3)}\right)$ and $lim_{x\to 3}\, (x-3)^2\left(\frac{9}{x^2(x-3)}\right)$ are finite. Thus $x = 3$ is an regular singular point.

x = 3
translate
Let u = x - 3
$y = \sum_{n=0}^{\infty} a_n u^{n+r}$

[20] 3.) Solve

$$y'' - 6y' + 9y = \frac{e^{3t}}{t}$$

Solve homogeneous equation: $y'' - 6y' + 9y = 0$.

Guess $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$

$r^2 - 6r + 9 = (r-3)^2 = 0$. Thus $r = 3$

Series Solutions Near a Regular Singular Point

MATH 365 *Ordinary Differential Equations*

J. Robert Buchanan

Department of Mathematics

Spring 2015

banach.millersville.edu/~bob/math365/Singular/main.pdf

Background

If P, Q, R polynomials

We will find a power series solution to the equation:

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

We will assume that t_0 is a **regular singular point**. This implies:

not ordinary

1. $P(t_0) = 0$,
2. $\lim_{t\to t_0} \dfrac{(t-t_0)Q(t)}{P(t)}$ exists,
3. $\lim_{t\to t_0} \dfrac{(t-t_0)^2R(t)}{P(t)}$ exists.

$\dfrac{Q(t_0)}{P(t_0)}$ or $\dfrac{R(t_0)}{P(t_0)}$ not defined

Simplification

If $t_0 \neq 0$ then we can make the change of variable $x = t - t_0$ and the ODE:

$$P(x+t_0)y'' + Q(x+t_0)y' + R(x+t_0)y = 0.$$

has a regular singular point at $x = 0$.

From now on we will work with the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

having a regular singular point at $x = 0$.

Assumptions (1 of 2)

Since the ODE has a regular singular point at $x = 0$ we can define

$$x\frac{Q(x)}{P(x)} = xp(x) \quad \text{and} \quad x^2\frac{R(x)}{P(x)} = x^2q(x)$$

which are analytic at $x = 0$ and

$$\lim_{x\to 0}\frac{xQ(x)}{P(x)} = \lim_{x\to 0} xp(x) = p_0$$

$$\lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0} x^2q(x) = q_0.$$

Assumptions (2 of 2)

Furthermore since $xp(x)$ and $x^2q(x)$ are analytic, near 0

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for all $-\rho < x < \rho$ with $\rho > 0$.

Re-writing the ODE

The second order linear homogeneous ODE can be written as

$$\begin{aligned}
0 &= P(x)y'' + Q(x)y' + R(x)y \\
&= y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y \\
&= x^2y'' + x^2\frac{Q(x)}{P(x)}y' + x^2\frac{R(x)}{P(x)}y \\
&= x^2y'' + x[xp(x)]y' + [x^2q(x)]y \\
&= x^2y'' + x[p_0 + p_1x + \cdots + p_nx^n + \cdots]y' \\
&\quad + [q_0 + q_1x + \cdots + q_nx^n + \cdots]y.
\end{aligned}$$

$y'' + p(x)y' + q(x)y = 0$

$x^2y'' + (xp(x))xy' + (x^2q(x))y = 0$

Euler $\quad x^2y'' + \alpha(x)xy' + \beta(x)y = 0$

$x = 0$ regular

$\Rightarrow \alpha(x) = xp(x)$

$\beta(x) = x^2q(x)$

are analy

Special Case: Euler's Equation

If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$\begin{aligned} 0 &= x^2y'' + x[p_0 + p_1x + \cdots + p_nx^n + \cdots]y' \\ &\quad + [q_0 + q_1x + \cdots + q_nx^n + \cdots]y \\ &= x^2y'' + p_0xy' + q_0y \end{aligned}$$

which is Euler's equation.

General Case

When $p_n \neq 0$ and/or $q_n \neq 0$ for some $n > 0$ then we will assume the solution to

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$$

has the form

$$\boxed{y(x) = x^r \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}}$$

an Euler solution multiplied by a power series.

Assume $a_0 \neq 0$

Guess

Solution Procedure

5.5 & 5.6

Assuming $y(x) = x^r \sum_{n=0}^{\infty} a_nx^n$ we must determine:

1. the values of r,
2. a recurrence relation for a_n,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_nx^n$.

← ordinary or regular singular

$$y'' + \frac{2}{4x}y' + \frac{1y}{4x} = 0$$

Example (1 of 8)

⇒ $x = 0$ is singular

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_nx^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

← Guess

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2} \end{aligned}$$

$$\lim_{x \to 0} x\left(\frac{2}{4x}\right) = \frac{1}{2}$$

$$\lim_{x \to 0} x^2\left(\frac{1}{4x}\right) = \lim_{x \to 0} \frac{x}{4} = 0$$

finite ⇒ regular singular

Example (2 of 8)

$$\begin{aligned} \boxed{0 = 4xy'' + 2y' + y} \\ &= 4x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2} + 2\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty} a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}4(r+n)(r+n-1)a_nx^{r+n-1} + \sum_{n=0}^{\infty}2(r+n)a_nx^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty} a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}[4(r+n)(r+n-1) + 2(r+n)]a_nx^{r+n-1} + \sum_{n=0}^{\infty} a_nx^{r+n} \end{aligned}$$

Example (3 of 8)

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}[4(r+n)(r+n-1) + 2(r+n)]a_nx^{r+n-1} + \sum_{n=0}^{\infty} a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=0}^{\infty} a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1} \end{aligned}$$

Example (4 of 8)

$n=0..\infty$ $\quad$ $n=1..\infty$

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1}\\
&= 2a_0r(2r-1)x^{r-1} + \sum_{n=1}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1}\\
&\quad + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1}\\
&= 2a_0r(2r-1)x^{r-1} + \sum_{n=1}^{\infty}\left[2a_n(r+n)(2r+2n-1)+a_{n-1}\right]x^{r+n-1}
\end{aligned}
$$

Example (5 of 8)

$$
\begin{aligned}
0 &= 2a_0r(2r-1)x^{r-1}\\
&\quad + \sum_{n=1}^{\infty}\left[2a_n(r+n)(2r+2n-1)+a_{n-1}\right]x^{r+n-1}
\end{aligned}
$$

$a_0 \neq 0$, $x \neq 0$ $\Rightarrow$ $\boxed{r(2r-1)=0}$

This implies

$$
\begin{aligned}
0 &= r(2r-1) \quad \text{(the indicial equation) and}\\
0 &= 2a_n(r+n)(2r+2n-1)+a_{n-1}
\end{aligned}
$$

$\Rightarrow r = 0, \frac{1}{2}$

Thus we see that $r=0$ or $r=\frac{1}{2}$ and the recurrence relation is

$$\boxed{a_n = -\frac{a_{n-1}}{(2r+2n)(2r+2n-1)}, \quad \text{for } n \geq 1.}$$

Solve for highest subscript

recurrence relation

Example, Case $r=0$ (6 of 8)

The recurrence relation becomes $a_n = -\dfrac{a_{n-1}}{2n(2n-1)}$.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2!}\\
a_2 &= -\frac{a_1}{(4)(3)} = \frac{a_0}{4!}\\
a_3 &= -\frac{a_2}{(6)(5)} = -\frac{a_0}{6!}\\
&\vdots\\
a_n &= \frac{(-1)^n a_0}{(2n)!}
\end{aligned}
$$

Thus $y_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{n+0} = a_0 \cos\sqrt{x}$.

Example, Case $r=1/2$ (7 of 8)

The recurrence relation becomes $a_n = -\dfrac{a_{n-1}}{(2n+1)2n}$.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!}\\
a_2 &= -\frac{a_1}{(5)(4)} = \frac{a_0}{5!}\\
a_3 &= -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!}\\
&\vdots\\
a_n &= \frac{(-1)^n a_0}{(2n+1)!}
\end{aligned}
$$

Thus $y_2(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n+1)!} x^{n+1/2} = a_0 \sin\sqrt{x}$.

Example (8 of 8)

We should verify that the general solution to

$$4xy'' + 2y' + y = 0$$

is

$$y(x) = c_1\cos\sqrt{x} + c_2\sin\sqrt{x}.$$

$$y = c_1\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n\right) + c_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}\right)$$

Remarks

- This technique just outlined will succeed provided $r_1 \neq r_2$ and $r_1 - r_2 \neq n \in \mathbb{Z}$.
- If $r_1 = r_2$ or $r_1 - r_2 = n \in \mathbb{Z}$ then we can always find the solution corresponding to the larger of the two roots r_1 or r_2.
- The second (linearly independent) solution will have a more complicated form involving $\ln x$.

General Case: Method of Frobenius

Given $x^2y'' + x[xp(x)]y' + \left[x^2q(x)\right]y = 0$ where $x = 0$ is a regular singular point and

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

are analytic at $x = 0$, we will seek a solution to the ODE of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

where $a_0 \neq 0$.

Substitute into the ODE

$$\begin{aligned}
0 &= x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} \\
&\quad + x\left[\sum_{n=0}^{\infty} p_n x^n\right] \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \left[\sum_{n=0}^{\infty} q_n x^n\right] \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} \\
&\quad + \left[\sum_{n=0}^{\infty} p_n x^n\right] \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \left[\sum_{n=0}^{\infty} q_n x^n\right] \sum_{n=0}^{\infty} a_n x^{r+n}
\end{aligned}$$

Collect Like Powers of x

$$\begin{aligned}
0 &= a_0 r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots \\
&\quad + (p_0 + p_1 x + \cdots)(a_0 r x^r + a_1(r+1)x^{r+1} + \cdots) \\
&\quad + (q_0 + q_1 x + \cdots)(a_0 x^r + a_1 x^{r+1} + \cdots) \\
&= a_0\left[r(r-1) + p_0 r + q_0\right]x^r \\
&\quad + \left[a_1(r+1)r + p_0 a_1(r+1) + p_1 a_0 r + q_0 a_1 + q_1 a_0\right]x^{r+1} \\
&\quad + \cdots \\
&= a_0\left[r(r-1) + p_0 r + q_0\right]x^r \\
&\quad + \left[a_1\left((r+1)r + p_0(r+1) + q_0\right) + a_0(p_1 r + q_1)\right]x^{r+1} \\
&\quad + \cdots
\end{aligned}$$

$$\cancel{a_0}\underbrace{\left[r(r-1) + p_0 r + q_0\right] = 0}_{\text{indicial eqn}}$$

Indicial Equation

If we define $F(r) = r(r-1) + p_0 r + q_0$ then the ODE can be written as

$$\begin{aligned}
0 &= a_0 F(r)x^r + \left[a_1 F(r+1) + a_0(p_1 r + q_1)\right]x^{r+1} \\
&\quad + \left[a_2 F(r+2) + a_0(p_2 r + q_2) + a_1(p_1(r+1) + q_1)\right]x^{r+2} \\
&\quad + \cdots
\end{aligned}$$

The equation

$$0 = F(r) = r(r-1) + p_0 r + q_0$$

is called the **indicial equation.** The solutions are called the **exponents of singularity.**

Recurrence Relation

The coefficients of x^{r+n} for $n \geq 1$ determine the **recurrence relation**:

$$\begin{aligned}
0 &= a_n F(r+n) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(r+k) + q_{n-k}\right) \\
a_n &= -\frac{\sum_{k=0}^{n-1} a_k\left(p_{n-k}(r+k) + q_{n-k}\right)}{F(r+n)}
\end{aligned}$$

provided $F(r+n) \neq 0$.

Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be r_1 and r_2.
- When r_1 and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$.
- Consequently the recurrence relation where $r = r_1$,

$$a_n(r_1) = -\frac{\sum_{k=0}^{n-1} a_k\left(p_{n-k}(r_1+k) + q_{n-k}\right)}{F(r_1+n)}$$

  is defined for all $n \geq 1$.
- One solution to the ODE is then

$$y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right).$$

Case: $r_1 - r_2 \notin \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$.
- Consequently the recurrence relation where $r = r_2$,

$$a_n(r_2) = -\frac{\sum_{k=0}^{n-1} a_k\left(p_{n-k}(r_2+k) + q_{n-k}\right)}{F(r_2+n)}$$

is defined for all $n \geq 1$.
- A second solution to the ODE is then

$$y_2(x) = x^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right).$$

Example

Find the indicial equation, exponents of singularity, and discuss the nature of solutions to the ODE

$$x^2y'' - x(2+x)y' + (2+x^2)y = 0$$

near the regular singular point $x = 0$.

Solution

$$\begin{aligned} p_0 &= \lim_{x\to 0} x\frac{-x(2+x)}{x^2} = -\lim_{x\to 0}(2+x) = -2 \\ q_0 &= \lim_{x\to 0} x^2\frac{2+x^2}{x^2} = \lim_{x\to 0}(2+x^2) = 2 \end{aligned}$$

The indicial equation is then

$$\begin{aligned} r(r-1) + (-2)r + 2 &= 0 \\ r^2 - 3r + 2 &= 0 \\ (r-2)(r-1) &= 0. \end{aligned}$$

The exponents of singularity are $r_1 = 2$ and $r_2 = 1$. Consequently we have one solution of the form

$$y_1(x) = x^2\left(1 + \sum_{n=1}^{\infty} a_n x^n\right).$$

repeated root

Case: $r_1 = r_2$ Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then $F(r) = (r - r_1)^2$.
- We have a solution to the ODE of the form

$$y_1(x) = x^r\left(1 + \sum_{n=1}^{\infty} a_n(r)x^n\right).$$

- Differentiating this solution and substituting into the ODE yields

$$\begin{aligned} 0 &= a_0F(r)x^r \\ &\quad + \sum_{n=1}^{\infty}\left[a_nF(r+n) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(r+k) + q_{n-k}\right)\right]x^{r+n} \\ &= a_0(r-r_1)^2x^r. \end{aligned}$$

when a_n solves the recurrence relation.

Case: $r_1 = r_2$ Equal Exponents of Singularity (2 of 4)

Recall: for the ODE $x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$ we can define the **linear operator**

$$L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y$$

so that the ODE can be written compactly as $L[y] = 0$.

Consider the infinite series solution to the ODE,

$$\phi(r,x) = x^r\left[1 + \sum_{n=1}^{\infty} a_n(r)x^n\right]. \quad = \quad \sum a_n x^{n+r}$$

Note: since the coefficients of the series depend on r we denote the solution as $\phi(r,x)$.

where choose $a_0 = 1$

Case: $r_1 = r_2$ Equal Exponents of Singularity (3 of 4)

$$\begin{aligned} 0 &= L[\phi](r_1,x) \\ 0 &= a_0(r-r_1)^2x^r\Big|_{r=r_1} \\ \frac{\partial}{\partial r}(0)\Big|_{r=r_1} &= \frac{\partial}{\partial r}\left(a_0(r-r_1)^2x^r\right)\Big|_{r=r_1} \\ 0 &= 2a_0(r-r_1)x^r\Big|_{r=r_1} + a_0(r-r_1)^2(\ln x)x^r\Big|_{r=r_1} \\ 0 &= a_0(r-r_1)^2(\ln x)x^r\Big|_{r=r_1} \\ 0 &= L\left[\frac{\partial\phi}{\partial r}\right](r_1,x) \end{aligned}$$

Thus a second solution to the ODE is $y_2(x) = \dfrac{\partial\phi(r,x)}{\partial r}\Big|_{r=r_1}$.

FYI

Don't assume $n, n+1 \Rightarrow n+2$

[20] 6.) Given the recursive relation $\boxed{a_{n+2} = 6a_{n+1} - 9a_n}$ where $a_0 = -1$ and $a_1 = 3$, prove that $a_n = 3^n(2n-1)$. You may use induction.

Proof by induction: First we prove that $a_n = 3^n(2n-1)$ for $n = 0, 1$:

Base case

$n = 0$: $3^0(2(0) - 1) = -1 = a_0$

$n = 1$: $3^1(2(1) - 1) = 3 = a_1$

Induction hypothesis: Suppose $a_k = 3^k(2k-1)$ for $k = n, n+1$.

Induction hyp

Then $a_n = 3^n(2n-1)$ and $a_{n+1} = 3^{n+1}(2(n+1)-1)$

Goal $\rightarrow$ $\boxed{\text{Claim: } a_{n+2} = 3^{n+2}(2(n+2)-1)}$

Prove Induct hyp $\Rightarrow$ $n+2$ case

$$
\begin{aligned}
a_{n+2} &= 6a_{n+1} - 9a_n \\
&= 6[3^{n+1}(2(n+1)-1)] - 9[3^n(2n-1)] \\
&= 2[3^{n+2}(2n+2-1)] - 3^{n+2}(2n-1) \\
&= 2[3^{n+2}(2n+1)] - 3^{n+2}(2n-1) \\
&= 3^{n+2}(4n+2) - 3^{n+2}(2n-1) \\
&= 3^{n+2}[4n+2-2n+1] \\
&= 3^{n+2}[2n+3] \\
&= 3^{n+2}[2(n+2)-1]
\end{aligned}
$$

Alternative answer (**not** covered in this class – see MATH:4050 Intro to Discrete Math):

Guess $a_n = x^n$. Then $a_{n+1} = x^{n+1}$ and $a_{n+2} = x^{n+2}$

Then $a_{n+2} = 6a_{n+1} - 9a_n$ implies $x^{n+2} - 6x^{n+1} + 9x^n = 0$.

Hence $x^n(x^2 - 6x + 9) = x^n(x-3)^2 = 0$. Thus $x = 3$

Claim: $a_n = c_1(3^n) + c_2(n3^n)$ satisfies $a_{n+2} - 6a_{n+1} + 9a_n = 0$

$$
\begin{aligned}
&c_1(3^{n+2}) + c_2((n+2)3^{n+2}) - 6[c_1(3^{n+1}) + c_2((n+1)3^{n+1})] + 9[c_1(3^n) + c_2(n3^n)] \\
&= c_1[3^{n+2} - 6(3^{n+1}) + 9(3^n)] + c_2[(n+2)3^{n+2} - 6((n+1)3^{n+1}) + 9(n3^n)] \\
&= c_1[3^{n+2} - 2(3^{n+2}) + (3^{n+2})] + c_2\{n[(3^{n+2} - 6(3^{n+1}) + 9(3^n)] + [(2)3^{n+2} - 6(3^{n+1})]\} \\
&= c_1[0] + c_2\{n[(3^{n+2} - 2(3^{n+2}) + (3^{n+2})] + [(2)3^{n+2} - 2(3^{n+2})]\} = 0
\end{aligned}
$$

Thus $a_n = c_1(3^n) + c_2(n3^n)$