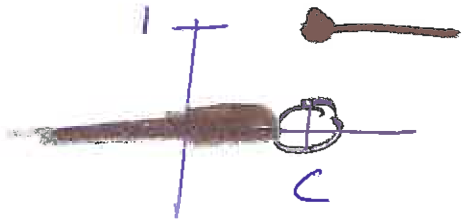
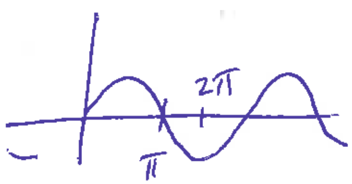
6.3: Step functions.

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Graph $u_c(t)$:

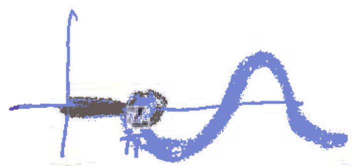


Graph $g(t) = sin(t)$.



$h(t) = \begin{cases} 0 & t < \pi \\ 1 \sin t & t \geq \pi \end{cases}$

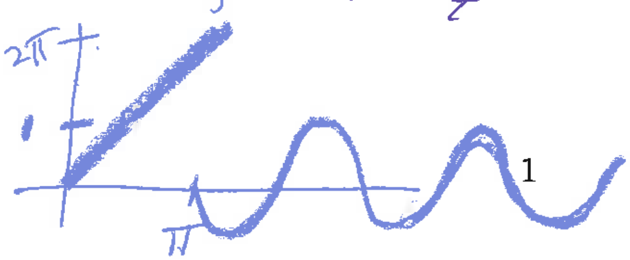
Graph $h(t) = u_\pi(t) sin(t)$.



Graph $f(t) = 2t + u_\pi(t)[sin(t) - 2t] = \begin{cases} 2t & t < \pi \\ \sin t & t \geq \pi \end{cases}$

$t < \pi$, $f(t) = 2t + 0$

$t \geq \pi$, $f(t) = \cancel{2t} + 1(\sin t - \cancel{2t}) = \sin t$



Example: $f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \leq t < 5; \\ f_3, & \text{if } 5 \leq t < 10; \\ f_4, & \text{if } t \geq 10; \end{cases}$

Hence

$$f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] + u_{10}(t)[f_4(t) - f_3(t)]$$

Check

Partial check:

If $t = 3$: $f(3) = f_1(3) + 0[f_2(3) - f_1(3)] + 0[f_3(3) - f_2(3)] + 0[f_4(3) - f_3(3)] = f_1(3)$

If $t = 9$: $f(9) = f_1(9) + 1[f_2(9) - f_1(9)] + 1[f_3(9) - f_2(9)] + 0[f_4(9) - f_3(9)] = f_3(9)$

Examples:

$f(t) = \begin{cases} 0 & 0 \leq t < 2 \\ t^2 & t \geq 2 \end{cases}$ implies $f(t) = 0 + u_2(t)[t^2 - 0]$

$= u_2(t)\, t^2$

$g(t) = \begin{cases} t^2 & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$ implies $g(t) = t^2 + u_3(t)[0 - t^2]$

$h(t) = \begin{cases} t & 0 \leq t < 4 \\ ln(t) & t \geq 4 \end{cases}$ implies $h(t) = t + u_4(t)[\ln t - t]$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

$$j(t) = \begin{cases} t & 0 \le t < 5 \\ 2 & 5 \le t \le 8 \\ e^t & t \ge 8 \end{cases} \quad \text{implies}$$

$$j(t) = t + u_5(t)[2-t] + u_8(t)[e^t - 2]$$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$ ← formula sheet

$$= e^{-cs}\mathcal{L}(g(t))$$

Let $g(t) = f(t+c) \Rightarrow f(t) = g(t-c)$

$g(t-c) = f(t-c+c) = f(t)$

$$\mathcal{L}(u_c(t)f(t)) = \mathcal{L}(u_c(t)g(t-c))$$

$$= e^{-cs}\mathcal{L}(g(t))$$

$$= e^{-cs}\mathcal{L}(f(t+c))$$

$$\boxed{\text{Formula 13: } \mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).}$$

Let $g(t) = f(t+c)$. Then $g(t-c) = f(t-c+c) = f(t)$. Thus

$$\mathcal{L}(u_c(t)f(t)) = \mathcal{L}(u_c(t)g(t-c)) = e^{-cs}\mathcal{L}(g(t))$$
$$= e^{-cs}\mathcal{L}(f(t+c)).$$

or equivalently

$$\boxed{\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).}$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Find the LaPlace transform of the following:

a.) $\mathcal{L}(u_3(t)(t^2-2t+1)) =$ $e^{-3s}F_1(s)$ ← what is $F_1(s)$

b.) $\mathcal{L}(u_4(t)(e^{-8t})) =$ $e^{-4s}F_2(s)$ ← what is $F_2(s)$

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ $e^{-2s}F_3(s)$ ← what is $F_3(s)$

Find the LaPlace transform of

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \ge 3 \end{cases} = u_3(t)e^{t-3}$

$$\mathcal{L}(u_3(t)e^{t-3}) = e^{-3s}\mathcal{L}(e^{t+3-3})$$

$$= e^{-3s}\mathcal{L}(e^t) = e^{-3s}\left(\frac{1}{s-1}\right)$$

$\cancel{\mathcal{L}(u_3(t)5 + u_4(t)(t-5)) = \mathcal{L}(5u_3(t)) + \mathcal{L}(u_4(t)\cdot(t-4))}$

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$ $\rightarrow \mathcal{L}(u_3(t)5 + u_4(t)[t-5-5]\}$

$= \mathcal{L}(u_3(t)\cdot 5) + \mathcal{L}(u_4(t)\cdot(t-10))$

$= e^{-3s}\mathcal{L}(5) + e^{-4s}\mathcal{L}(t+4-10)$

$=$ ~

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$.

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$

$= \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$

where $f(t) = \mathcal{L}^{-1}(F(s))$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) =$ __________

b.) $\mathcal{L}^{-1}(e^{4s}\frac{1}{s^2-3}) =$ __________

c.) $\mathcal{L}^{-1}(e^{-s}\frac{5}{(s-3)^4}) =$ __________

d.) $\mathcal{L}^{-1}(\frac{e^s}{4s}) =$ __________

e.) $\mathcal{L}^{-1}(e^s) =$ __________

f.) $\mathcal{L}^{-1}(e^s\frac{1}{(s-3)^2+4}) =$ __________

g.) $\mathcal{L}^{-1}(e^s\frac{2s-5}{s^2+6s+13}) =$ __________