Quiz 6 Math 2560 Sect 81 — Show your work

Dec 2, 2016

[3] 1.) Circle T for True and F for false:

1a.) If ϕ is a solution to a first order **linear homogeneous** differential equation, then $c\phi$ is also a solution to this equation. T

1b.) If ϕ is a solution to a first order linear differential equation, then $c\phi$ is also a solution to this equation. F

[4] 2a.) Solve: $\mathbf{x}' = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \mathbf{x}$

Find eigenvalues:

$$\begin{vmatrix} 1-r & 5 \\ 2 & 3-r \end{vmatrix} = (1-r)(3-r) - 10 = r^2 - 4r + 3 - 10 = r^2 - 4r - 7 = 0.$$

$$r = \frac{4 \pm \sqrt{16-4(-7)}}{2} = \frac{4 \pm \sqrt{4(4+7)}}{2} = \frac{4 \pm 2\sqrt{11}}{2} = 2 \pm \sqrt{11}.$$

1 positive e. value
1 negative e. value
$\Rightarrow$ saddle
unstable

$2 + \sqrt{11} > 0$ and $2 - \sqrt{11} < 0$. Thus the critical point $\mathbf{x} = \mathbf{0}$ is an unstable saddle point.

Find eigenvectors:

$$\begin{bmatrix} 1-(2 \pm \sqrt{11}) & 5 \\ 2 & 3-(2 \pm \sqrt{11}) \end{bmatrix} = \begin{bmatrix} -1 \mp \sqrt{11} & 5 \\ 2 & 1 \mp \sqrt{11} \end{bmatrix}$$

$$\begin{bmatrix} -1 \mp \sqrt{11} & 5 \\ 2 & 1 \mp \sqrt{11} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \pm \sqrt{11} \end{bmatrix} = \begin{bmatrix} 5(-1 \mp \sqrt{11}) + 5(1 \pm \sqrt{11}) \\ 10 + (1 \mp \sqrt{11})(1 \pm \sqrt{11}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check

Thus $\begin{bmatrix} 5 \\ 1 \pm \sqrt{11} \end{bmatrix}$ is an eigenvector with eigenvalue $2 \pm \sqrt{11}$.

Answer: $\mathbf{x} = c_1 \begin{bmatrix} 5 \\ 1+\sqrt{11} \end{bmatrix} e^{(2+\sqrt{11})t} + c_2 \begin{bmatrix} 5 \\ 1-\sqrt{11} \end{bmatrix} e^{(2-\sqrt{11})t}$

slope

[2] 2b.) Determine if the critical point is stable, asymptotically stable, or unstable.

unstable

slope

[1] 2c.) Draw the direction field and sketch a few trajectories.

Graph $x_2 = \frac{1+\sqrt{11}}{5} x_1$ and $x_2 = \frac{1-\sqrt{11}}{5} x_1$ for $x_1 > 0$ and $x_1 < 0$ along with some other trajectories.

$r = 2 + \sqrt{11} > 0$ $r = 2 - \sqrt{11} < 0$

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ two real e. values

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

nodal source
unstable

Case 1b: $r_1 < r_2 < 0$

nodal sink
asym stable
$\lim_{t \to \infty} X(t) = 0$

Case 1c: $r_2 < 0 < r_1$

saddle
unstable

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ 1 repeated

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ 2 complex e. value

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is $\vec{X} = c_1 (\vec{V}$ cos

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

unstable
spiral source

Case 3a: $\lambda < 0$

asy stable
spiral sink

Case 3a: $\lambda = 0$

stable
center