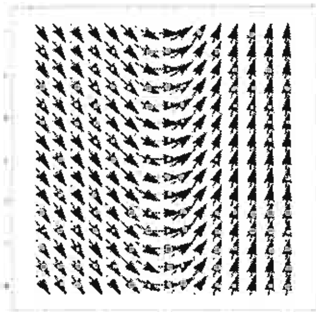
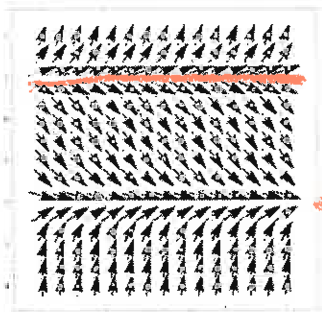
Problems 17 - 20 show the slope field for a first order differential equations. In addition to determining and classifying all equilibrium solutions (if any), also draw the trajectories satisfying the initial values $y(0) = 1$, $y(1) = 0$, $y(1) = 2$, $y(0) = -3$.

17.)



18.)

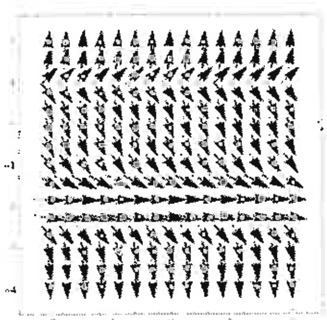


$y' = f(y)$

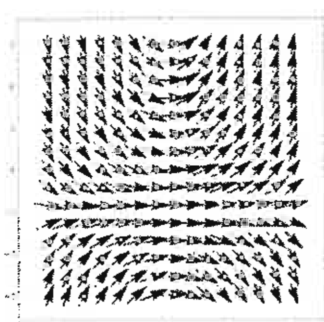
← unstable

← stable

19.)

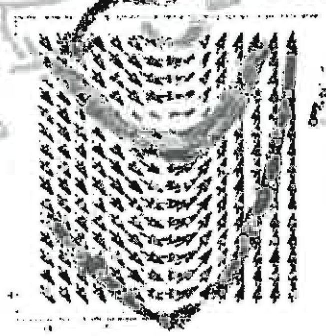


20.)



Problems 17 - 20 show the slope field for a first order differential equations. In addition to determining and classifying all equilibrium solutions (if any), also draw the trajectories satisfying the initial values $y(0) = 1$, $y(1) = 0$, $y(1) = 2$, $y(0) = -3$.

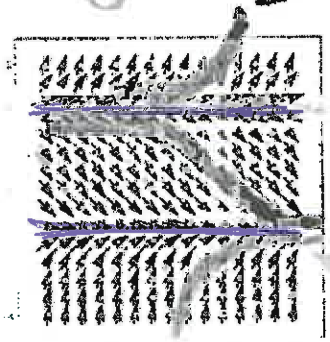
17.)



No equilibrium solution.

$y(1) = 2$

18.)



$y(0) = 1$,

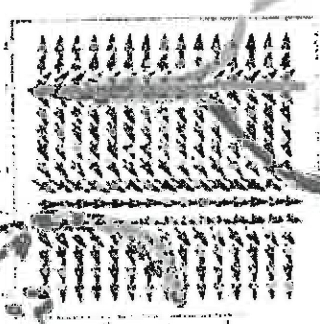
$y = 1$ is unstable. $y = -2$ is asymptotically stable.

$y(1) = 0$

$y(0) = -3$

$y(1) = 2$

19.)



$y(0) = 1$

$y = 1$ is unstable. $y = -2$ is semi-stable.

$y(1) = 0$

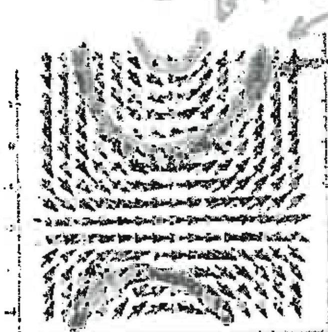
$y(0) = -3$

$y(1) = 2$

$y(0) = 1$

$y(1) = 0$

20.)

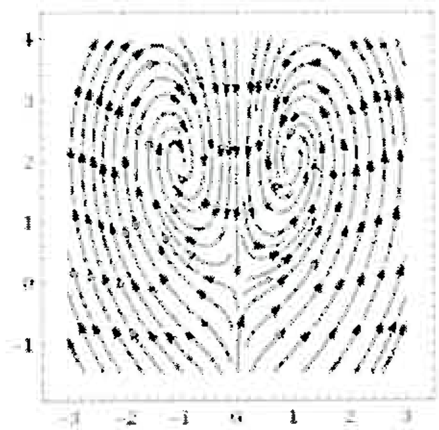


$y = -2$ is unstable.

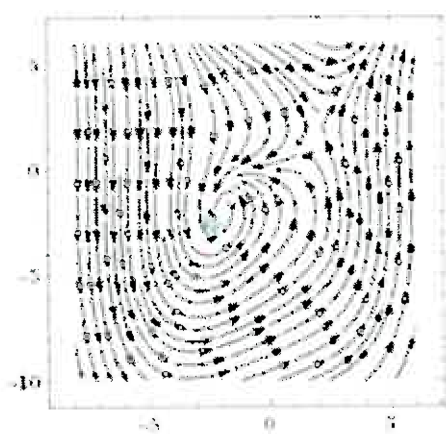
$y(0) = -3$

Problems 21-23 show the stream plot in the $x_1 - x_2$-plane for a system of two first order differential equations In addition to determining and classifying all equilibrium solutions, also draw the trajectories satisfying the initial values $(x_1(0), x_2(0)) = (0, 1)$, $(x_1(0), x_2(0)) = (1, 0)$, $(x_1(0), x_2(0)) = (1, 2)$, $(x_1(0), x_2(0)) = (-1, 0)$. Also describe the basins of attraction.

21.)

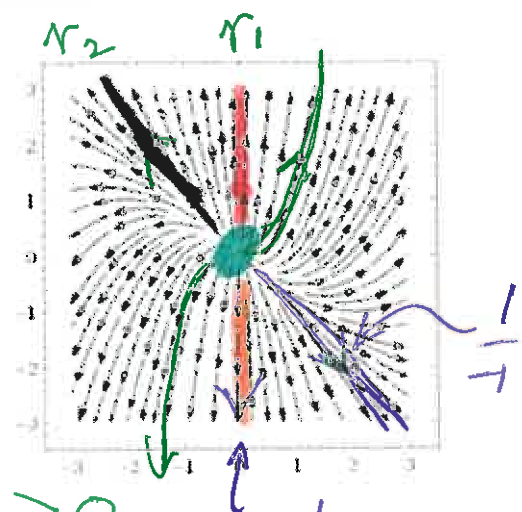


22.)



9.1

23.)



$$x' = AX$$

$$\boxed{x = 0 \quad \text{equilibrium}}$$

Unstable

$r_1 > r_2 > 0$

slope $\frac{1}{0}$

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{r_2 t}$$

Solve $\vec{x}\,' = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & -6 \\ 2 & 1 & -3 \end{bmatrix} \vec{x}$

(row 2 = 3 • row 1
$\Rightarrow$ det = 0
$\Rightarrow$ 0 is an e. value

① Find e. values

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & -2 \\ 0 & 3-\lambda & -6 \\ 2 & 1 & -3-\lambda \end{vmatrix}$$

$$= (-1)^{1+1}(-\lambda)\begin{vmatrix} 3-\lambda & -6 \\ 1 & -3-\lambda \end{vmatrix} + (-1)^{1+2}(0)\begin{vmatrix} 1 & -2 \\ 1 & -3-\lambda \end{vmatrix}$$

$$+ (-1)^{1+3}(2)\begin{vmatrix} 1 & -2 \\ 3-\lambda & -6 \end{vmatrix}$$

$$= +(-\lambda)[-9 + \lambda^2 + 6] - 0 + 2[\cancel{-6} + \cancel{6} - 2\lambda]$$

$$= -\lambda^3 + 3\lambda - 4\lambda = -\lambda^3 - \lambda$$

$$= -\lambda(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda = 0 \qquad \lambda^{\cancel{2}} = \pm\sqrt{-1} = \pm i$$

② Find e. vectors

$\lambda = 0$: Solve $(A - 0I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \\ 2 & 1^{-1} & -3^{+2} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2/2 & 0 & -1/2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 - \tfrac{1}{2} x_3 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 = x_3 \end{array}\right\} \quad \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} = \begin{bmatrix} 1/2 \\ 2 \\ 1 \end{bmatrix} x_3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} e^{0t} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$ is a soln to $\vec{x}\,' = A\vec{x}$

$r = i:$

$$\left[\begin{array}{ccc|c} -i & 1 & -2 & 0 \\ 0 & 3-i & -6 & 0 \\ 2 & 1 & -3-i & 0 \end{array}\right]$$

$i \times \text{row } 1$

$$\left[\begin{array}{ccc|c} 1 & i & -2i & 0 \\ 0 & (3-i)^{(3+i)} & (-6)^{(3+i)} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(3+i) \times \text{row } 2$

$$\left[\begin{array}{ccc|c} 1 & i - \frac{i}{10}(10) & -2i - \frac{i}{10}(-18-6i) & 0 \\ 0 & 1.0 & -1.8-.6i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.2i-0.6 & 0 \\ 0 & 1.0 & -1.8-0.6i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.6+0.2i \\ 1.8+0.6i \\ 1 \end{bmatrix} x_3$$

Let $x_3 = 5$:
$$\begin{bmatrix} 3+1i \\ 9+3i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} + i \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$r = -i \Rightarrow \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} - i \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is an e. vector

$\boxed{\text{complex conjugate}}$

General soln

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + c_2 \left(\begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \sin t \right)$$

$$+ c_3 \left(\begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \cos t \right)$$

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ ← 7.8

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

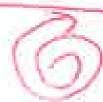
7.6

Suppose eigenvector corresponding to eigenvalue is

$$\begin{pmatrix} v_1 \pm i w_1 \\ v_2 \pm i w_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \pm i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right)$$
$$+ c_2 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right)$$

Case 3a: $\lambda > 0$  or  7.6

Case 3a: $\lambda < 0$  or 

Case 3a: $\lambda = 0$ 

7.6: Complex eigenvalue example: Solve $\mathbf{x}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \mathbf{x}$

Step 1 Find eigenvalues: $det(A - rI) = 0$

$$det(A - rI) = \begin{vmatrix} 3-r & -13 \\ 5 & 1-r \end{vmatrix} = (3-r)(1-r) + 65 = r^2 - 4r + 68 = 0$$

Thus $r = \frac{4 \pm \sqrt{4^2 - 4(68)}}{2} = \frac{4 \pm \sqrt{4(4-68)}}{2} = \frac{4 \pm 2\sqrt{-64}}{2} = 2 \pm 8i$

Step 2 Find eigenvectors: Solve $(A - rI)\mathbf{x} = \mathbf{0}$

$$A - (2 \pm 8i)I = \begin{bmatrix} 3-(2 \pm 8i) & -13 \\ 5 & 1-(2 \pm 8i) \end{bmatrix} = \begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix}$$

Solve $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix} = \begin{bmatrix} (1 \mp 8i)13 - 13(1 \mp 8i) \\ 5(13) + (-1 \mp 8i)(1 \mp 8i) \end{bmatrix} = \begin{bmatrix} 0 \\ 65 + (-1 + 64i^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus any non-zero multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ is an eigenvector of A with eigen value $2 \pm 8i$.

Note: $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ since $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus we can use either $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ or any nonzero multiple.

General solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} cos(8t) - \begin{bmatrix} 8 \\ 0 \end{bmatrix} sin(8t) \right) + c_2 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} sin(8t) + \begin{bmatrix} 8 \\ 0 \end{bmatrix} cos(8t) \right)$$

Slope field for x_2 vs x_1: $\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{x_2'}{x_1'} = \frac{3x_1 - 13x_2}{5x_1 + x_2}$

Note slope 0's occur when $3x_1 - 13x_2 = 0$, ie, $x_2 = \frac{13}{3}x_1$.

Note slope ∞'s occur when $5x_1 + x_2 = 0$, ie, $x_2 = -5x_1$.

Determine where slopes are positive vs negative for regions between these lines.

For example, along the x_2 axis slope is negative: $x_1 = 0$ and $\frac{dx_2}{dx_1} = \frac{-13x_2}{x_2} = \frac{-13}{1}$

For example, along the x_1 axis slope is positive: $x_2 = 0$ and $\frac{dx_2}{dx_1} = \frac{3x_1}{5x_1} = \frac{3}{5}$

But don't need to be accurate