3.6 = 4.4 Variation of Parameters

$$\text{Solve } ay''' + by'' + cy' + ey = g(t)$$

1) Find homogeneous solutions:

Suppose general homogeneous soln to $ay''' + by'' + cy' + ey = 0$ is

$$y = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6 = 4.4: $y = u_1(t)\phi_1 + u_2(t)\phi_2 + u_3(t)\phi_3$ **and solve for u_1, u_2, and u_3**

3 unknown functions, thus might want 3 equations.

1 equation will come from plugging into $ay''' + by'' + cy' + ey = g(t)$

We will choose the remaining equations so that we avoid u_i''

$y = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ implies

$$y' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2 + u_3\phi_3' + u_3'\phi_3$$

I.e., $y' = u_1\phi_1' + u_2\phi_2' + u_3\phi_3' + u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3$

Avoid 2nd derivative u_i'': Choose $u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 = 0$

Thus $y' = u_1\phi_1' + u_2\phi_2' + u_3\phi_3'$

$y'' = u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3'' + u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3'$ by product rule.

Avoid 2nd derivative u_i'': Choose $u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' = 0$

Thus $y'' = u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3''$

$y''' = u_1\phi_1''' + u_2\phi_2''' + u_3\phi_3''' + u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3''$ by product rule.

Plug $y = u_1(t)\phi_1 + u_2(t)\phi_2 + u_3(t)\phi_3$ (using above derivatives including simplifications) into $ay''' + by'' + cy' + ey = 0$

After lots of work, this simplifies to $u_1'\phi_1'; + u_2'\phi_2'' + u_3'\phi_3'' = \frac{g(t)}{a}$

Thus we have 3 unknowns (the functions, u_1, u_2, u_3) and 3 equations (the two equations we choose in order to avoid u_i'') as well as the equation we get by plugging in $y = u_1(t)\phi_1 + u_2(t)\phi_2 + u_3(t)\phi_3$ into DE and doing lots of simplification. These 3 equations are

$$u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 = 0$$
$$u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' = 0$$
$$u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' = \frac{g(t)}{a}$$

In matrix form: $$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{g(t)}{a} \end{pmatrix}$$

Solve using Cramer's rule.

NOTE: The coefficient matrix is the related to the Wronskian (same matrix, but before you take the determinant). Thus if $W =$ the coefficient matrix, $\mathbf{u} = n \times 1$ vector with ith coordinate u_i and $\mathbf{b} = n \times 1$ vector where all coordinates are 0 except the last term is $\frac{g(t)}{a}$, then the matrix formula can be written as

$$W\mathbf{u}' = \mathbf{b}$$

Don't forget to integrate to find u_i after using Cramer's rule to find u_i'.

Note: If you are likely to forget to divide by a, you might want to do so in the beginning (after solving homogeneous). This is what the book does:

$$y''' + \frac{b}{a}y'' + \frac{c}{a}y' + \frac{e}{a}y = \frac{g(t)}{a}$$