4.1: General Theory of nth Order Linear Equations

Thm: $L(y) = y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y$ is a linear function.

Proof: Let a, b be real numbers.

$L(af + bg)$

$= (af + bg)^{(n)} + p_1(t)(af + bg)^{(n-1)} + ... + p_{n-1}(t)(af + bg)' + p_n(t)(af + bg)$

$= af^{(n)} + bg^{(n)} + p_1(af^{(n-1)} + bg^{(n-1)}) + ... + p_{n-1}(af' + bg') + p_n(af + bg)$

$= af^{(n)} + p_1 af^{(n-1)} + ... + p_{n-1}af' + p_n af + bg^{(n)} + p_1 bg^{(n-1)} + ... + p_{n-1}bg' + p_n bg$

$= a[f^{(n)} + p_1 f^{(n-1)} + ... + p_{n-1}f' + p_n f] + b[g^{(n)} + p_1 g^{(n-1)} + ... + p_{n-1}g' + p_n g]$

$= aL(f) + bL(g)$

Theorem: If ϕ_i, $i = 1, ..., n$ are solutions to a homogeneous linear differential equation (i.e., $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$ (*)) , then $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to this linear differential equation.

Pf: Since ϕ_i, $i = 1, ..., n$ are solutions to (*), $L(\phi_i) = 0$ for $i = 1, ..., n$.
Thus $L(c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n) = c_1L(\phi_1) + c_2L(\phi_2) + ... + c_nL(\phi_n) = 0$.
Thus $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to (*).

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1cos(t) + k_2sin(t)$. Then $y' = -k_1sin(t) + k_2cos(t)$

$y(0) = -1$: $\quad -1 = k_1cos(0) + k_2sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $\quad -3 = -k_1sin(0) + k_2cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0,$

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t)$ is a solution to this IVT Then

$$y(t_0) = y_0: \quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + ... + c_n\phi_n(t_0)$$

$$y'(t_0) = y_1: \quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + ... + c_n\phi_n'(t_0)$$

$$\vdots$$

$$y^{(n-1)}(t_0) = y_{n-1}: \quad y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0)$$

To find IVP solution, need to solve above system of equations for the unknowns c_i, $i = 1, ..., n$.

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the c_i.

Note that in these equations the c_i are the unknowns and the $y_i, \phi_i(t_0), ...\phi_i^{(n-1)}(t_0)$, are the constants.
We can translate this linear system of equations into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Definition: The Wronskian of the functions, ϕ_1, ϕ_2,..., ϕ_n is

$$W(\phi_1, \phi_2, ..., \phi_n) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Theorem: Suppose that ϕ_i, $i = 1, ..., n$ are solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, then there is a unique choice of constants c_i such that $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}$.

Recall $\phi_1, ..., \phi_n$ are linearly independent iff $c_1 = ... = c_n = 0$ is the only solution to $c_1\phi_1 + ... + c_n\phi_n = \mathbf{0}$.

If ϕ_i are functions of t, then $\mathbf{0}$ is the constant function, $\mathbf{0}(t) = 0$ for all t. Thus $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ for all t.

Hence $c_1\phi_1^{(k)}(t) + ... + c_n\phi_n^{(k)}(t) = 0$ for all t, k if derivatives exist.

Thus $\phi_1, ..., \phi_n$ are linearly independent iff for any given f, $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

iff the following system of equations has a unique solution

$$\begin{gathered} c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) = 0 \\ c_1\phi_1'(t) + c_2\phi_2'(t) + ... + c_n\phi_n'(t) = 0 \\ \vdots \\ c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + ... + c_n\phi_n^{(n-1)}(t) = 0 \end{gathered}$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$,

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1: Solve $c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0$

Or equivalently, solve $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

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nth order LINEAR differential equation:

Theorem 4.1.1: If $p_i : (a, b) \to R$, $i = 1, ..., n$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

Claim: If p_i are continuous on (a, b), if $\phi_1, ..., \phi_n$ are linearly independent solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0,$$

then $\{\phi_1, ..., \phi_n\}$ is a basis for the solution set to this differential equation.

Theorem 4.1.2: If p_i are continuous on (a, b), suppose that ϕ_i, $i = 1, ..., n$ are solutions to $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$.
If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as

$$y = c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n \text{ for some constants } c_i.$$

Defn: The $\phi_1, ..., \phi_n$ are called a fundamental set of solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

Theorem: Given any nth order homogeneous linear differential equation, there exist a set of n functions which form a fundamental set of solutions.