Linear Algebra Review: Determinants

Defn: $detA = \Sigma \pm a_{1j_1}a_{2j_2}...a_{nj_n}$

2×2 short-cut: $det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

3×3 short-cut: $det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

Definition of Determinant using cofactor expansion

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$. Then expanding along row i,

$$detA = \Sigma_{k=1}^{n}(-1)^{i+k}a_{ik}detA_{ik}.$$

Or expanding along column j,

$$detA = \Sigma_{k=1}^{n}(-1)^{k+j}a_{kj}detA_{kj}.$$

Properties of Determinants

Thm: If $A \xrightarrow[R_i \to cR_i]{} B$, then $detB = c(detA)$.

Thm: If $A \xrightarrow[R_i \leftrightarrow R_j]{} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow[R_i + cR_j \to R_i]{} B$, then $detB = detA$.

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $detA = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Thm: A square matrix is invertible if and only if

$$detA \neq 0.$$

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $detA \neq 0$.

Linear Algebra Review: Cramer's Rule.

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with $\mathbf{b}$.

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $detA \neq 0$. Then

$$x_i = \frac{detA_i(\mathbf{b})}{detA}.$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$$

$$det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$$

$$det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$$

Thus $x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = \frac{9}{2}$.

3.6 Variation of Parameters

$$\text{Solve } ay'' + by' + cy = g(t)$$

1) Find homogeneous solutions:

Suppose general homogeneous soln to $ay'' + by' + cy = 0$ is

$$y = c_1\phi_1(t) + c_2\phi_2(t)$$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)\phi_1 + u_2(t)\phi_2$ **and solve for** u_1 **and** u_2

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} \frac{g(t)}{a} dt = -\int \frac{\phi_2(t)g(t)}{aW(\phi_1,\phi_2)} dt$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} \frac{g(t)}{a} dt = \int \frac{\phi_1(t)g(t)}{aW(\phi_1,\phi_2)} dt$$

Note that in both cases, you divide by the Wronskian:

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

General non-homogeneous solution :

$$y = c_1\phi_1(t) + c_2\phi_2(t) + (u_1(t)\phi_1 + u_2(t)\phi_2)$$

Note: If you are likely to forget to divide by a when finding u_1 and u_2, then when finding non-homogeneous solution, divide both sides of DE by a so that you have $1y'' + py' + qy = \frac{g(t)}{a}$ (this is what our textbook does).

Note: These formulas generalize to higher order linear DEs. To find u_i replace the i^{th} column of the Wronskian with a column of all zeros except the last entry which is 1 and follow 2nd order case.

Note: ORDER MATTERS. Once you choose an ordering of your homogeneous solutions $\phi_1, ...\phi_n$ for an nth order DE, be consistent with that order.

Note: If you are solving an IVP, you must find the complete general solution including any non-homogeneous parts BEFORE using your initial values to solve for the c_i's.

3.6 Variation of Parameters

$$\text{Solve } y'' - 2y' + y = e^t ln(t)$$

1) Find homogeneous solutions:

Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2 e^{rt}$, and

$r^2 e^{rt} - 2re^{rt} + e^{rt} = 0$ implies $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1 e^t + c_2 te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$ $\blacksquare$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^{2t} + te^{2t} - te^{2t} = e^{2t}$$

$$\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix} = -\phi_2 = -te^t \qquad \begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix} = \phi_1 = e^t$$

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{W(\phi_1, \phi_2)} \frac{g(t)}{a} dt = -\int \frac{(te^t)(e^t ln(t))}{e^{2t}} dt$$

$$= -\int tln(t)dt = -[\frac{t^2 ln(t)}{2} - \int \frac{t}{2} dt] = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$$

Integration by parts: $\begin{array}{cc} u = ln(t) & dv = tdt \\ du = \frac{dt}{t} & v = \frac{t^2}{2} \end{array}$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t) dt = \int \frac{(e^t)(e^t ln(t))}{e^{2t}} dt$$

$$= \int ln(t)dt = [tln(t) - \int dt] = tln(t) - t$$

Integration by parts: $\begin{array}{cc} u = ln(t) & dv = dt \\ du = \frac{dt}{t} & v = t \end{array}$

General solution :

$$y = c_1 e^t + c_2 te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$$

which simplifies to $y = c_1 e^t + c_2 te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2 e^t$