Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$. Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0)$, $y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \quad \text{implies} \quad \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if $det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) Wronskian of $cos(t)$, $sin(t) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix} = cos^2(t) + sin^2(t) = 1 > 0$.

2.) Wronskian of $e^{dt}cos(nt)$, $e^{dt}sin(nt) = \begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$

$= e^{dt}cos(nt)[de^{dt}sin(nt) + ne^{dt}cos(nt)] - e^{dt}sin(nt)[de^{dt}cos(nt) - ne^{dt}sin(nt)]$

$= e^{2dt}(cos(nt)[dsin(nt) + ncos(nt)] - sin(nt)[dcos(nt) - nsin(nt)])$

$= e^{2dt}(dcos(nt)sin(nt) + ncos^2(nt)] - dsin(nt)cos(nt) + nsin^2(nt)])$

$= e^{2dt}(ncos^2(nt) + nsin^2(nt)) = ne^{2dt}(cos^2(nt) + sin^2(nt)) = ne^{2dt} > 0$ for all t.