Existence and Uniqueness for LINEAR DEs.

Homogeneous:

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = 0$$

Non-homogeneous: $g(t) \neq 0$

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = g(t)$$

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

Partial proof: $y = \phi_1(t)$ is a solution to $y' + p(t)y = 0$ implies

Thus $y = c\phi_1(t)$ is a solution to $y' + p(t)y = 0$ since

$y = \psi(t)$ is a solution to $y' + p(t)y = g(t)$ implies

Thus $y = c\phi_1(t) + \psi(t)$ is a solution to $y' + p(t)y = g(t)$ since

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0,$$
$$y'(t_0) = y'_0$$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where p and q are functions of t (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$.

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to
$ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $\quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $\quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0)$, $y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{c} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) W$(cos(t), sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) W$(e^{dt}cos(nt), e^{dt}sin(nt)) =$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(nt$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$$

$$= ne^{2dt}[cos^2(nt) + sin^2(nt)] \; = \; ne^{2dt} > 0 \text{ for all } t.$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$.
There is a unique choice of constants c_1 and c_2 such that $c_1\phi_1 + c_2\phi_2$ satisfies this homog linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_0'$.

iff

$W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0.$

Thm 3.2.4: Given the hypothesis of thm 3.2.1, suppose that ϕ_1 and ϕ_2 are two solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Defn If ϕ_1 and ϕ_2 satisfy the conditions in thm 3.2.4, then ϕ_1 and ϕ_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

FYI: Linear Independence and the Wronskian

Defn: ϕ_1 and ϕ_2 are linearly dependent if there exists constants c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$ and

$$c_1\phi_1(t) + c_2\phi_2(t) = 0 \text{ for all } t \in (a, b)$$

Thm 3.3.1: If $\phi_1 : (a, b) \to R$ and $\phi_2(a, b) \to R$ are differentiable functions on (a, b) and
if $W(\phi_1, \phi_2)(t_0) \neq 0$ for some $t_0 \in (a, b)$, then
ϕ_1 and ϕ_2 are linearly independent on (a, b).
Moreover, if ϕ_1 and ϕ_2 are linearly dependent on (a, b), then $W(\phi_1, \phi_2)(t) = 0$ for all $t \in (a, b)$

Proof idea:

If $c_1\phi_1(t) + c_2\phi_2(t) = 0$ for all $t \in (a, b)$,
then $c_1\phi_1'(t) + c_2\phi_2'(t) = 0$ for all $t \in (a, b)$

Solve the following linear system of equations for c_1, c_2

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) = 0$$
$$c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = 0$$

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words the fundamental set of solutions $\{\phi_1, \phi_2\}$ $\blacksquare$ to $y'' + p(t)y' + q(t)y = 0$ form a basis for the set of all solutions to this linear homogeneous DE.