2.3: Modeling with differential equations.

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t) =$ amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t) =$ amount of salt in tank in grams.

Note $Q(0) = 5$ g

rate in $=$ (2 liters/min)($Q(t) \cdot tsin(t^2)$ g/liters)

$$= 2Qtsin(t^2) \text{ g/min}$$

rate out $=$ (2 liters/min)$\left(\frac{Q(t)g}{4\text{liters}}\right) = \frac{Q}{2}$ g/min

$\frac{dQ}{dt} =$ rate in - rate out $= 2Qtsin(t^2) - \frac{Q}{2}$

$\frac{dQ}{dt} = Q(2tsin(t^2) - \frac{1}{2})$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

$$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \frac{1}{2})dt = \int 2tsin(t^2)dt - \int \frac{1}{2}dt$$

Let $u = t^2$, $du = 2tdt$

$$ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C$$
$$= -cos(t^2) - \frac{t}{2} + C$$

$$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$$

$$Q = Ce^{-cos(t^2) - \frac{t}{2}}$$

$Q(0) = 5 : \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

Thus $Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$

Long-term behaviour:

$$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.