Lecture 5: Triangulations & simplicial complexes (and cell complexes).

in a series of preparatory lectures for the Fall 2013 online course MATH: 7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Target Audience: Anyone interested in **topological data analysis** including graduate students, faculty, industrial researchers in bioinformatics, biology, business, computer science, cosmology, engineering, imaging, mathematics, neurology, physics, statistics, etc.

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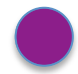
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

Building blocks for a simplicial complex

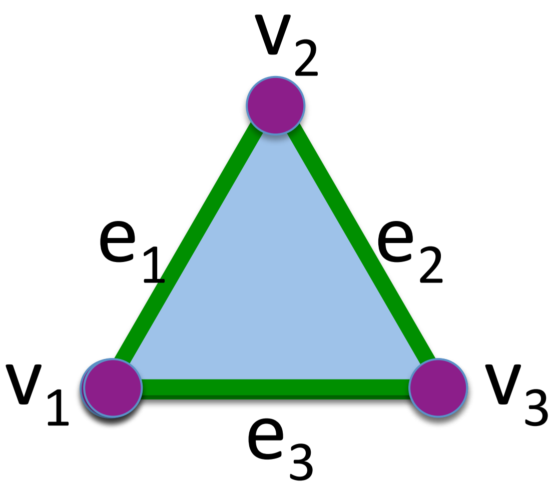
0-simplex = vertex = v



1-simplex = edge = $\{v_1, v_2\}$

Note that the boundary of this edge is $v_2 + v_1$

2-simplex = triangle = $\{v_1, v_2, v_3\}$



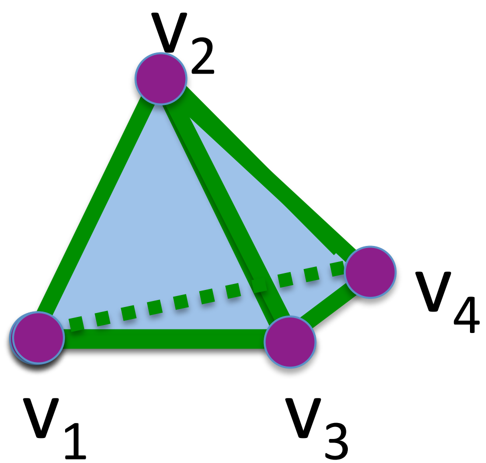
Note that the boundary of this triangle is the cycle

$$e_1 + e_2 + e_3$$

$$= \{v_1, v_2\} + \{v_2, v_3\} + \{v_1, v_3\}$$

Building blocks for a simplicial complex

3-simplex = $\{v_1, v_2, v_3, v_4\}$ = tetrahedron



boundary of $\{v_1, v_2, v_3, v_4\}$ =
$\{v_1, v_2, v_3\} + \{v_1, v_2, v_4\} + \{v_1, v_3, v_4\} + \{v_2, v_3, v_4\}$

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n-simplex = $\{v_1, v_2, \ldots, v_{n+1}\}$

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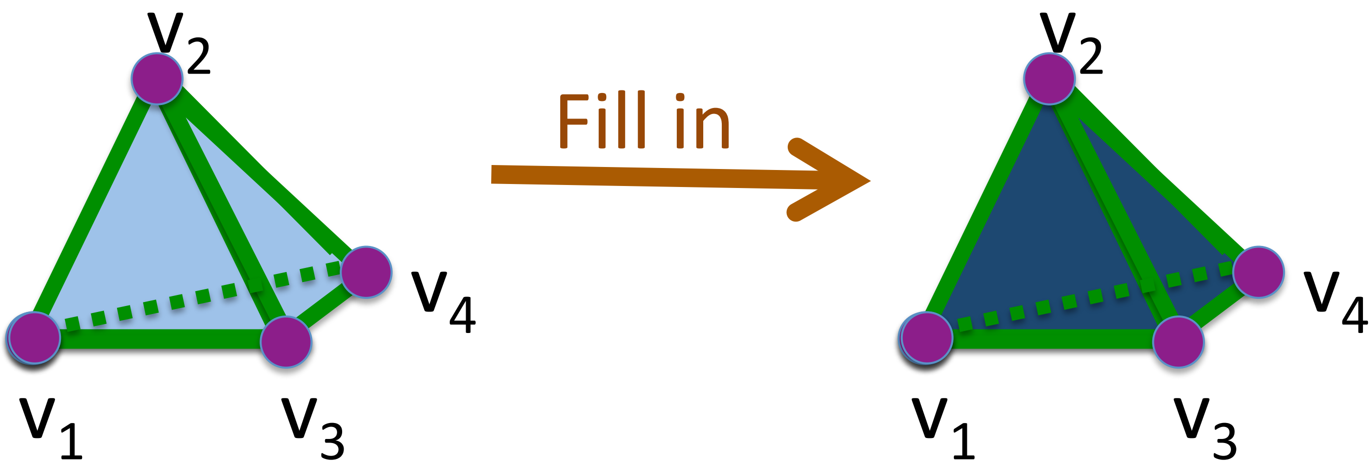

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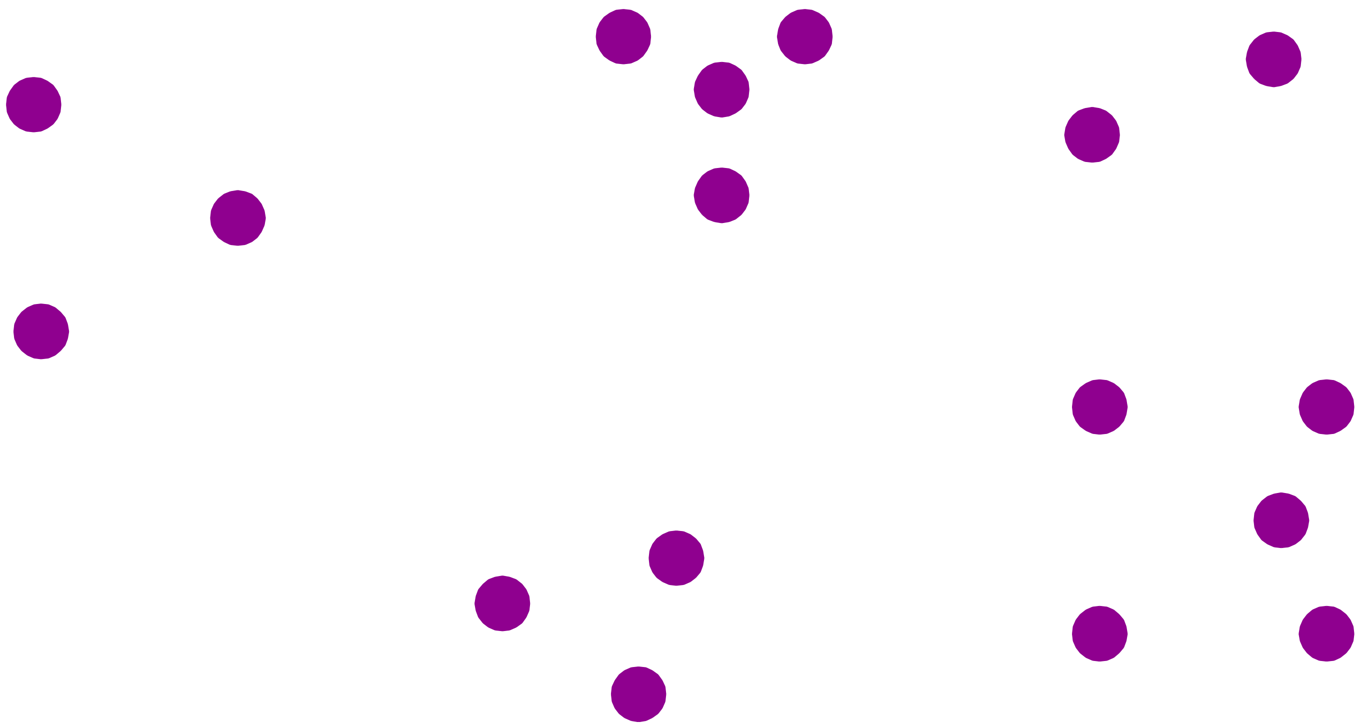
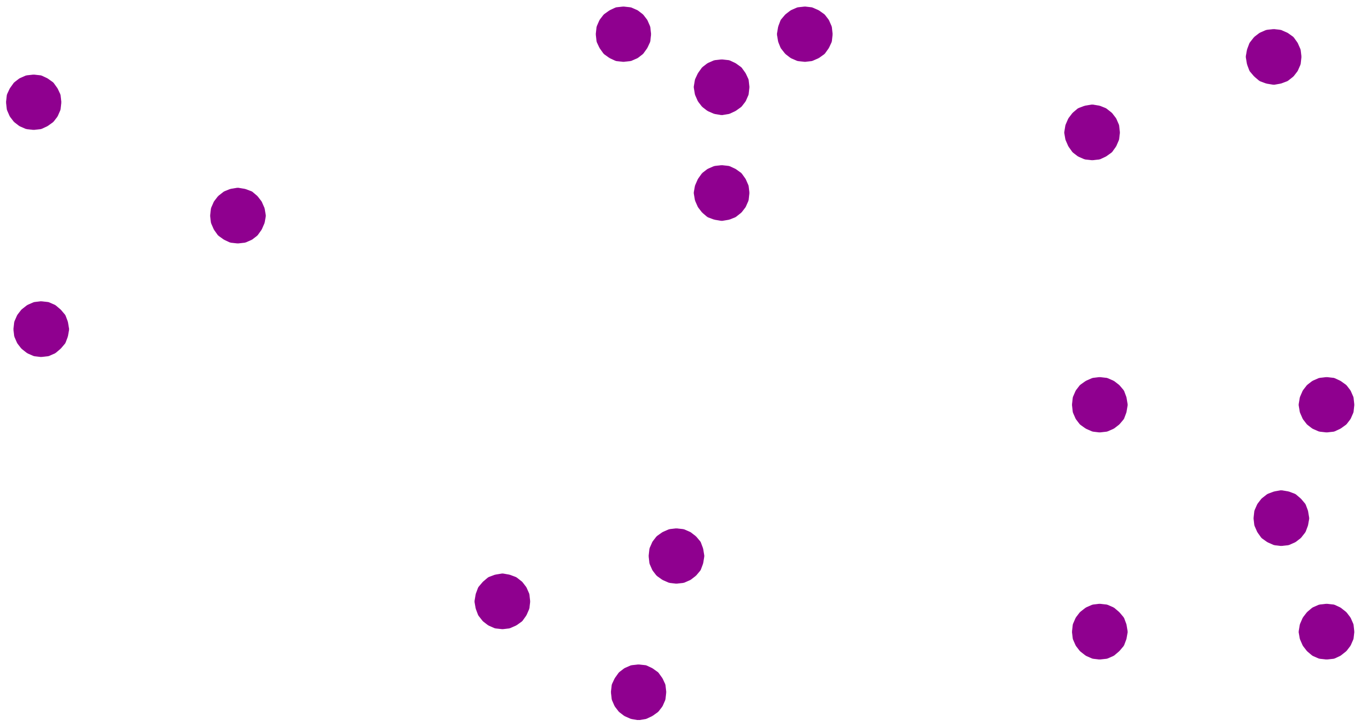
Creating a simplicial complex



0.) Start by adding 0-dimensional vertices (0-simplices)

Creating a simplicial complex

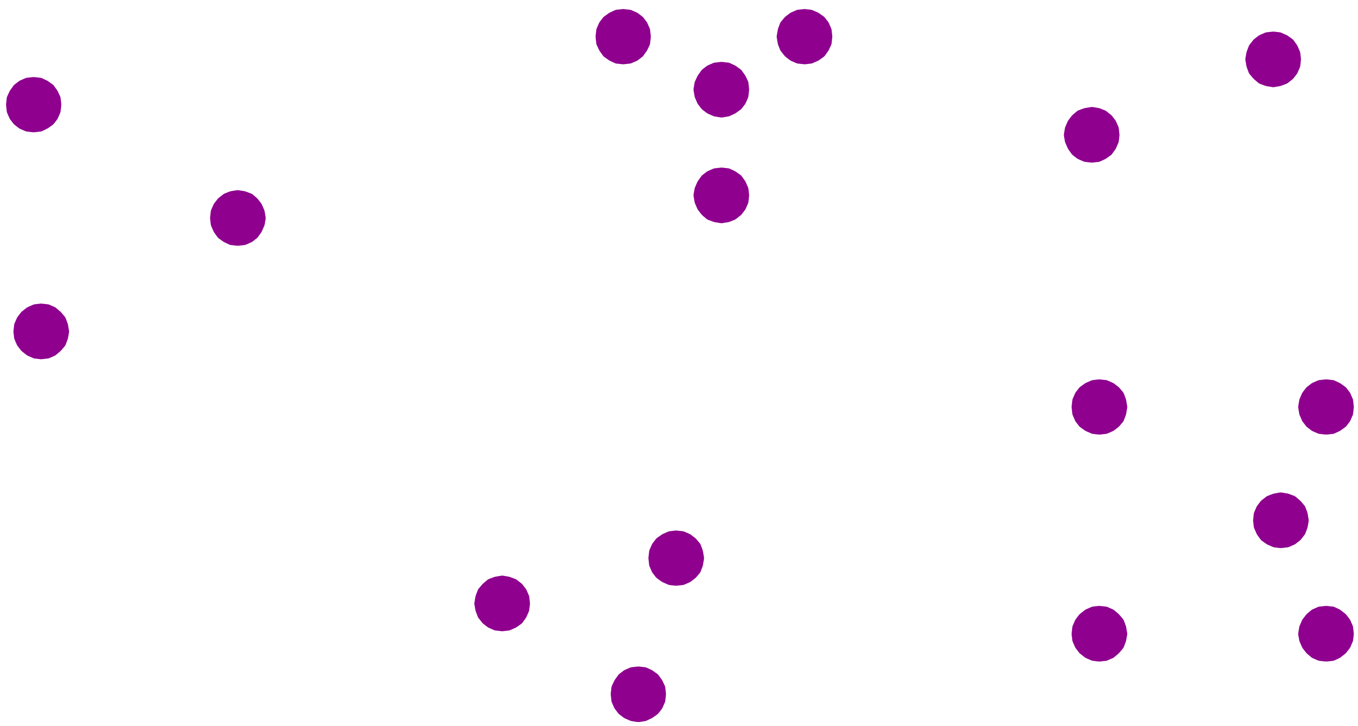


1.) Next add 1-dimensional edges (1-simplices).

Note: These edges must connect two vertices.

I.e., the boundary of an edge is two vertices

Creating a simplicial complex



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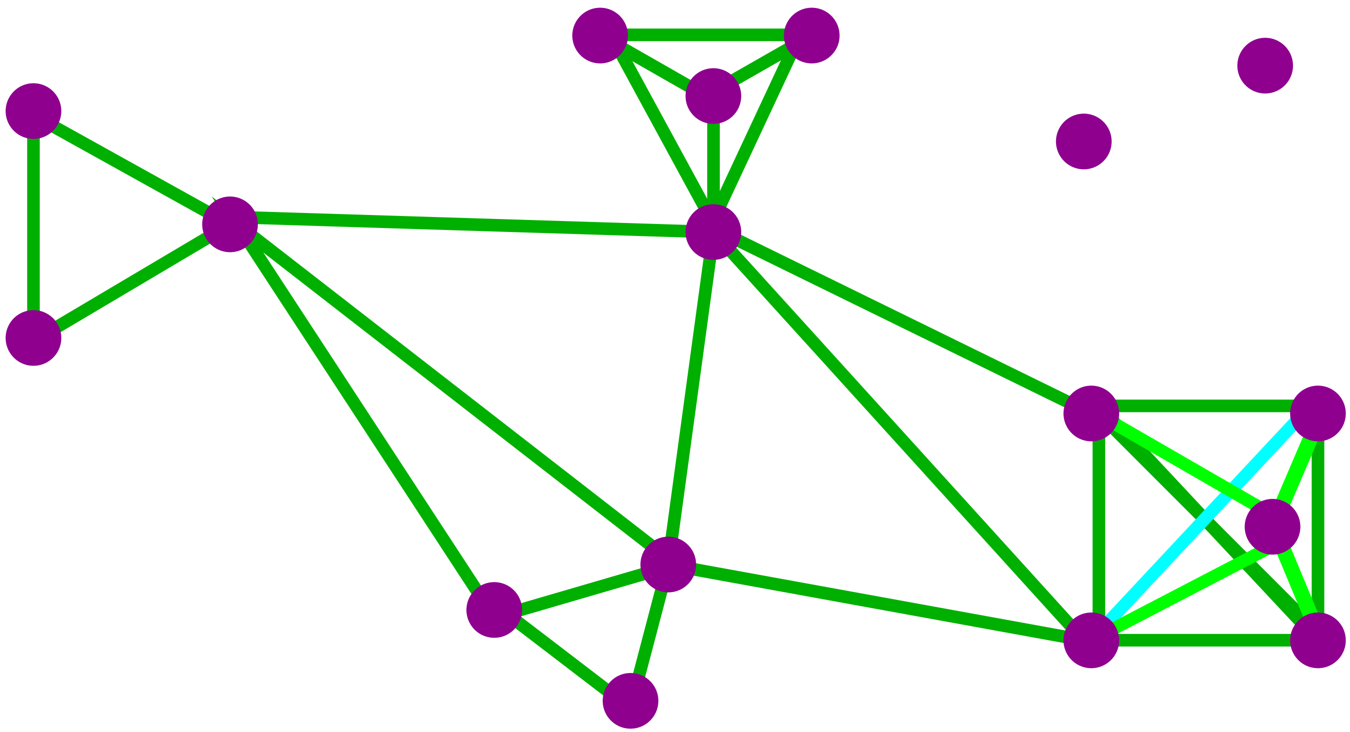

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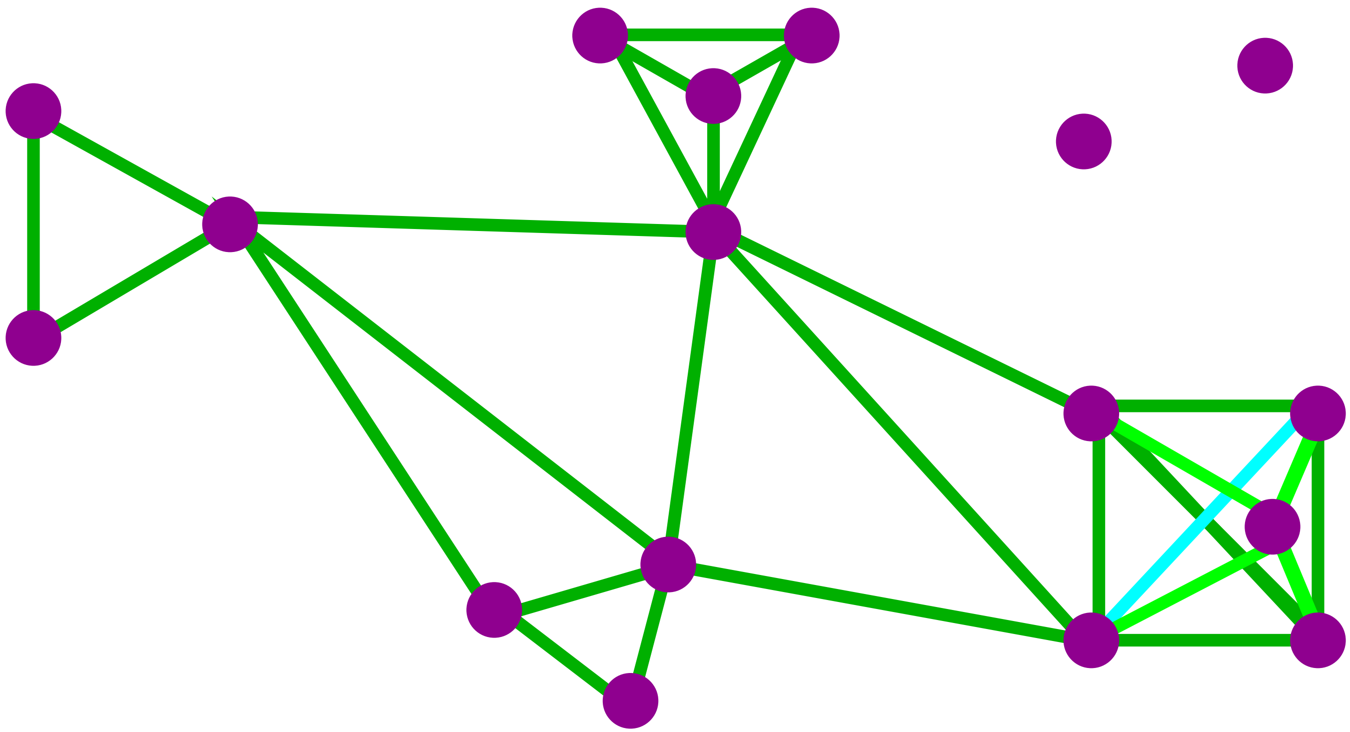
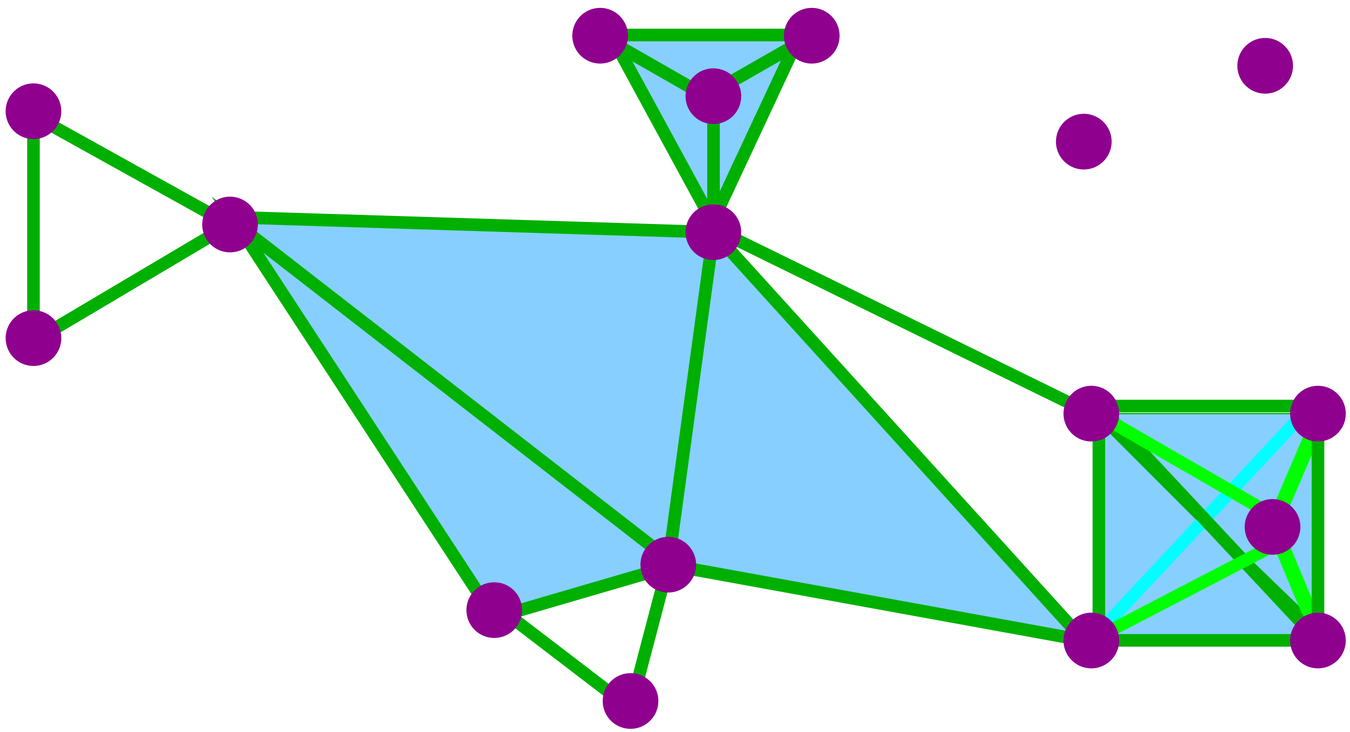


2.) Add 2-dimensional triangles (2-simplices).

Boundary of a triangle = a cycle consisting of 3 edges.

Creating a simplicial complex



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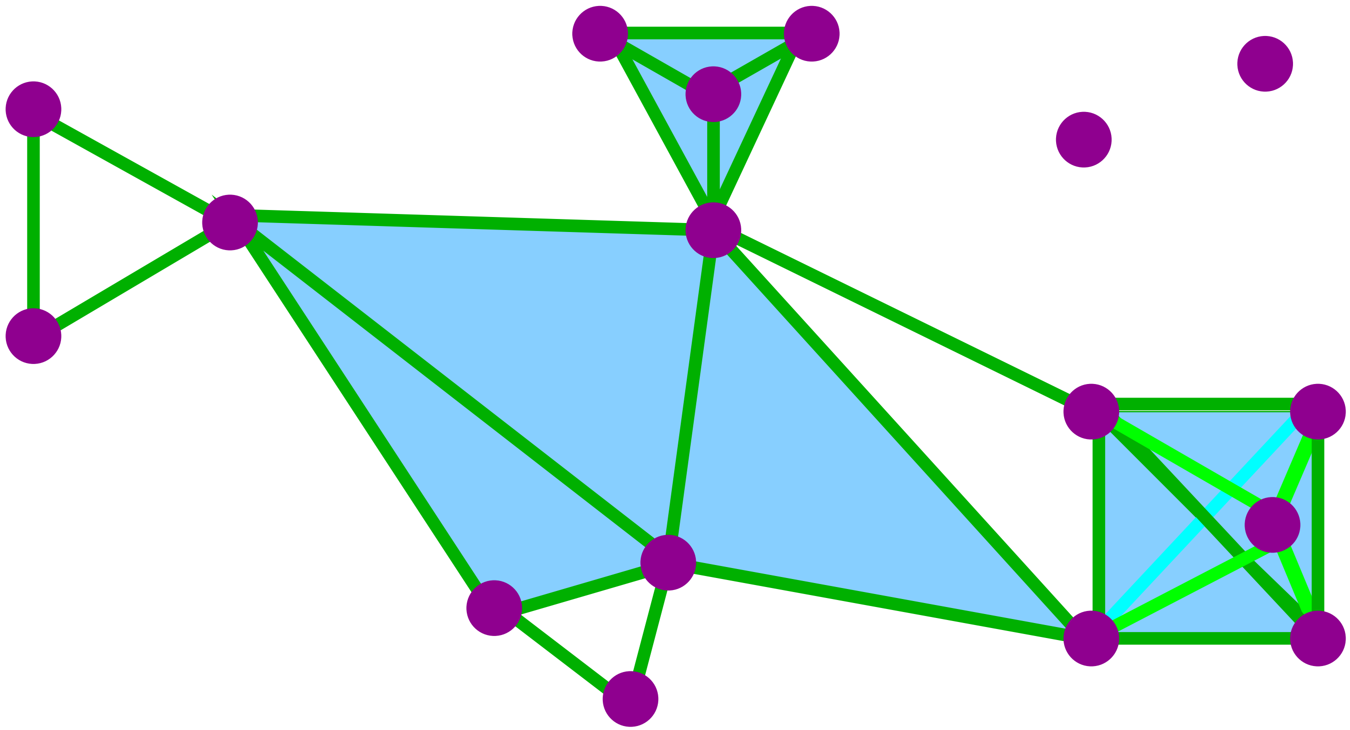
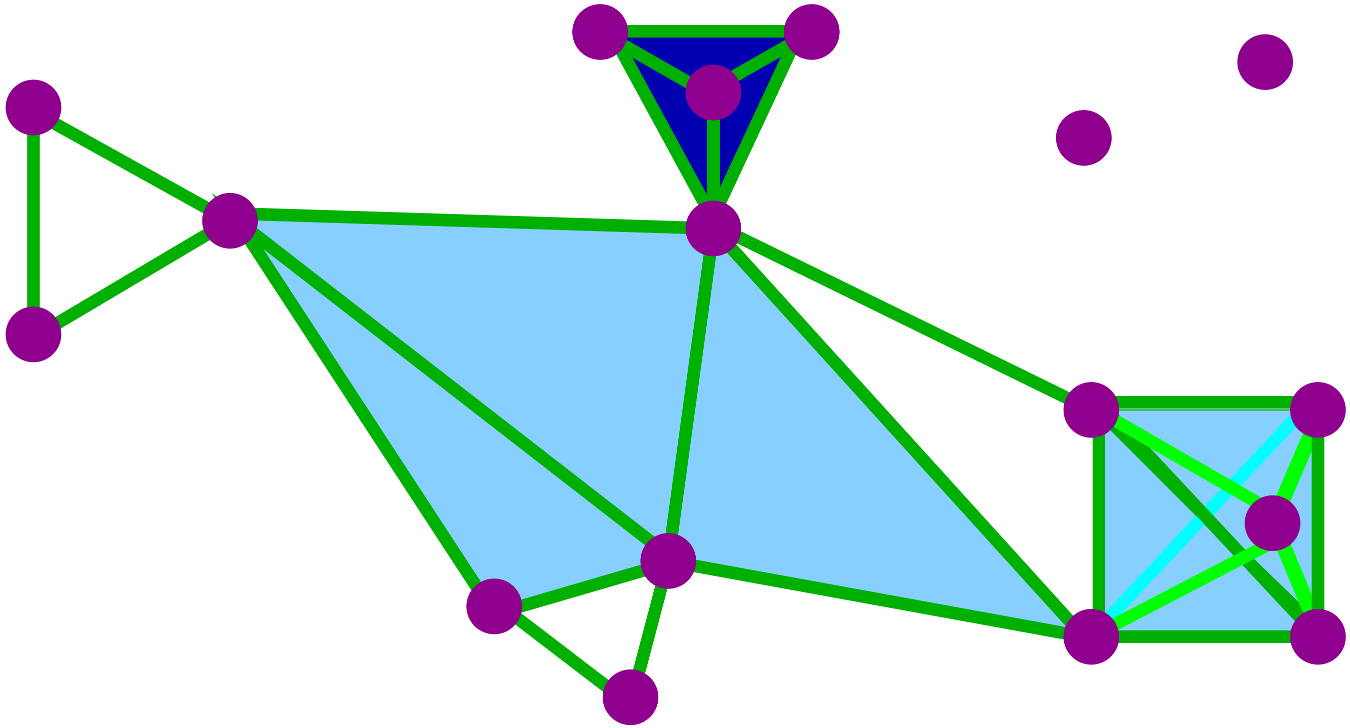


3.) Add 3-dimensional tetrahedrons (3-simplices).

Boundary of a 3-simplex

= a cycle consisting of its four 2-dimensional faces.

Creating a simplicial complex



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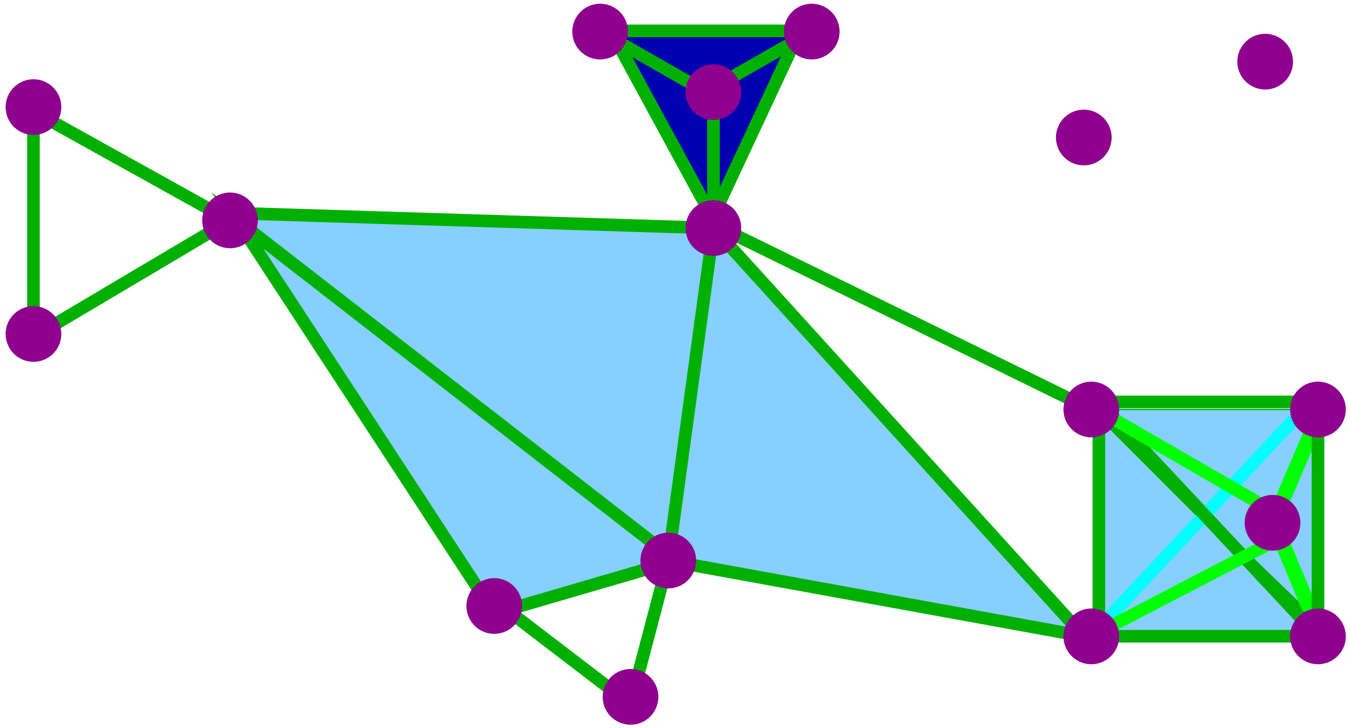


4.) Add 4-dimensional 4-simplices, $\{v_1, v_2, \ldots, v_5\}$.

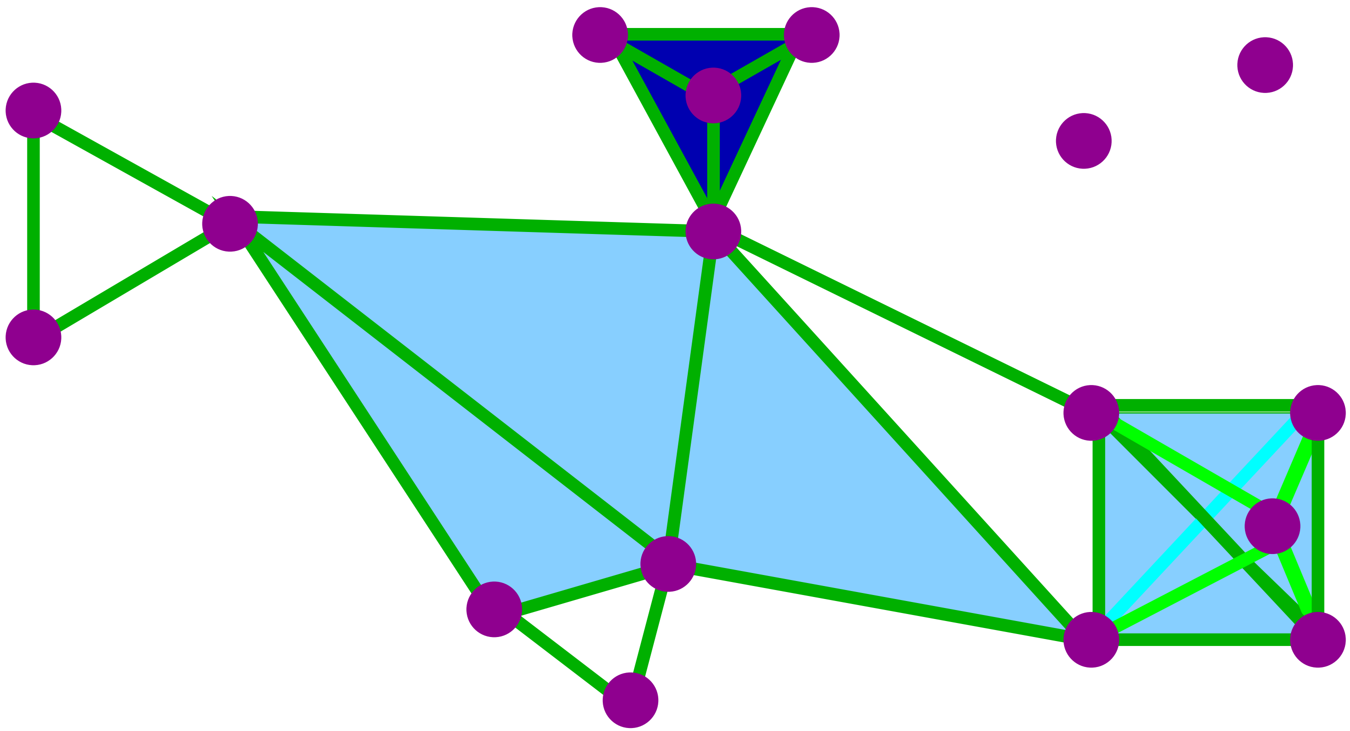
Boundary of a 4-simplex

= a cycle consisting of 3-simplices.

$= \{v_2, v_3, v_4, v_5\} + \{v_1, v_3, v_4, v_5\} + \{v_1, v_2, v_4, v_5\}$

$+ \{v_1, v_2, v_3, v_5\} + \{v_1, v_2, v_3, v_4\}$

Creating a simplicial complex



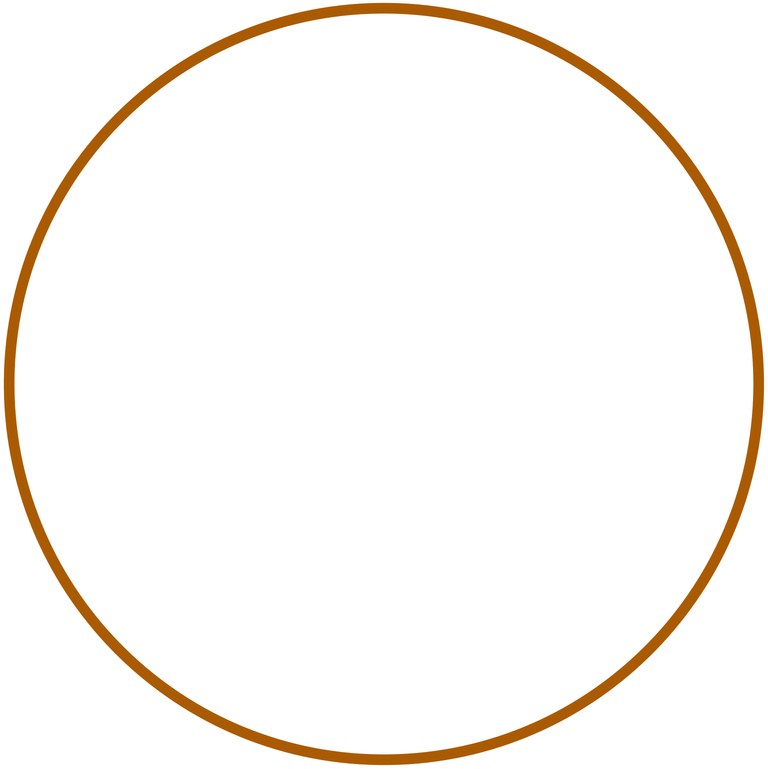
n.) Add n-dimensional n-simplices, $\{v_1, v_2, \ldots, v_{n+1}\}$.

Boundary of a n-simplex

= a cycle consisting of (n-1)-simplices.

Example: Triangulating the circle.

circle = { x in R^2 : $||x|| = 1$ }



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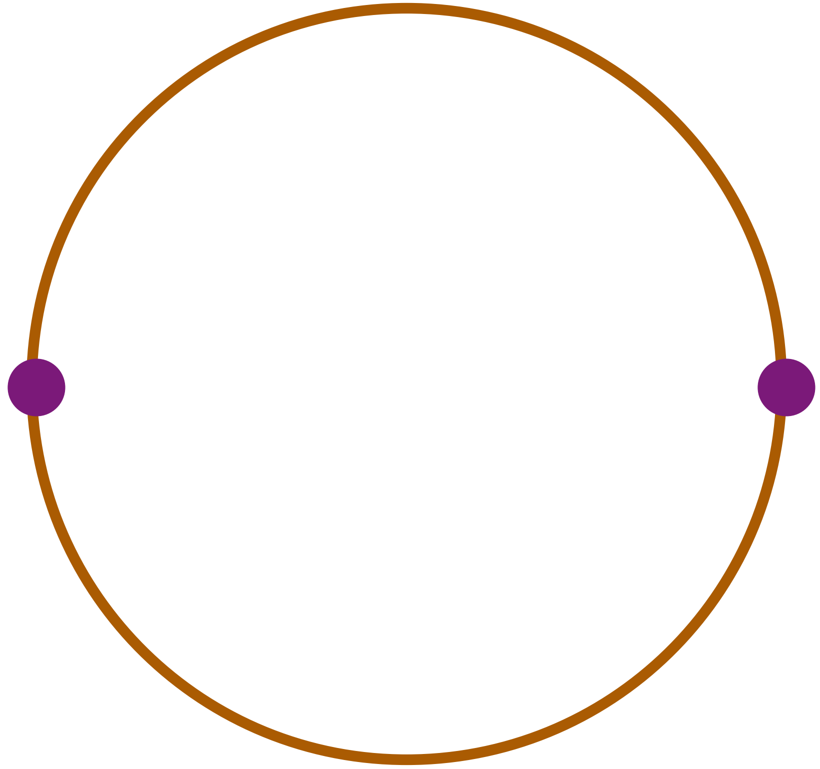

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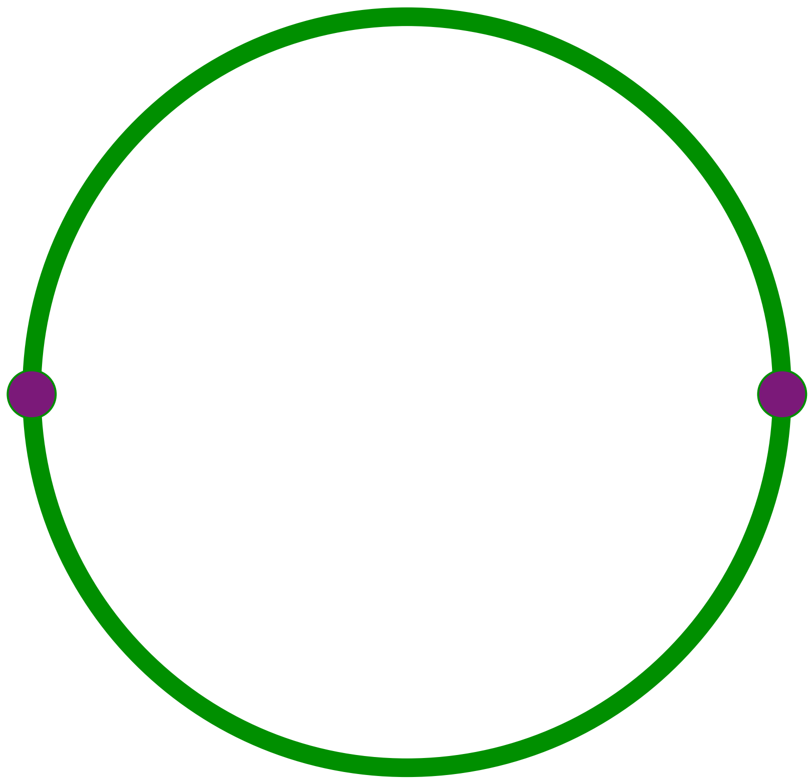

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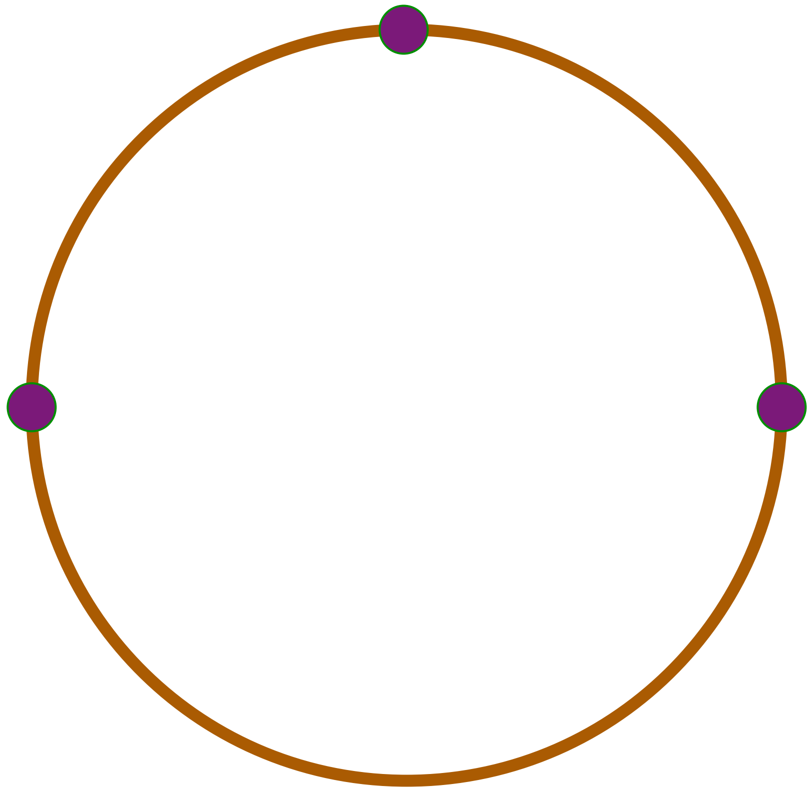

Example: Triangulating the circle.

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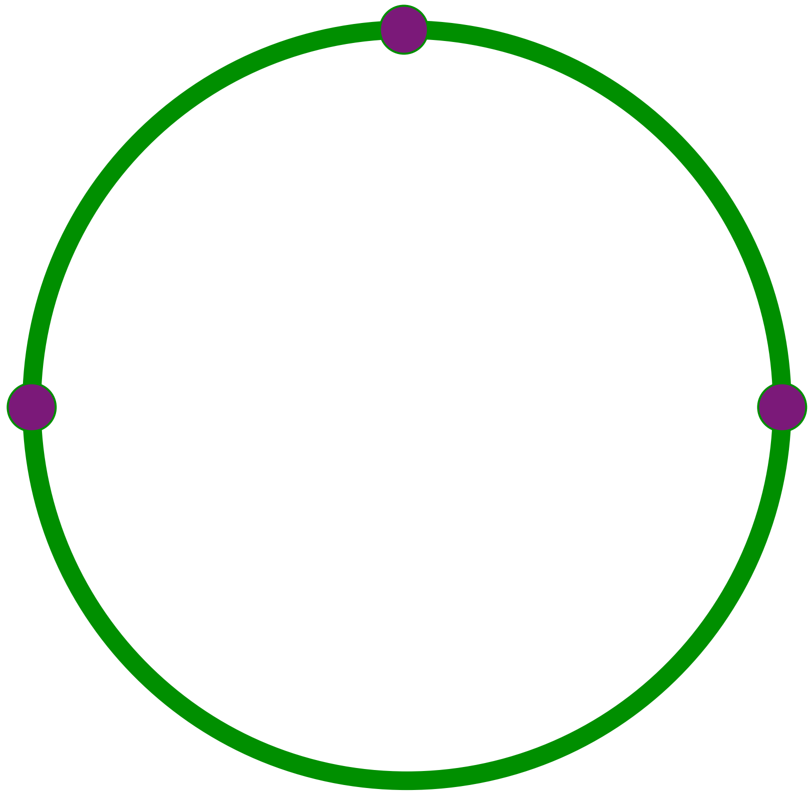

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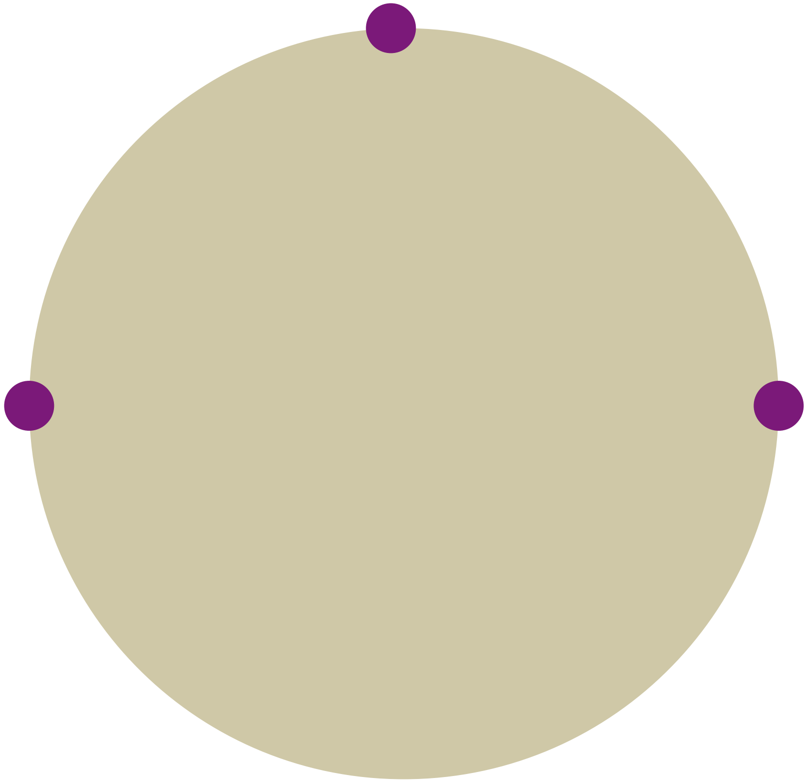

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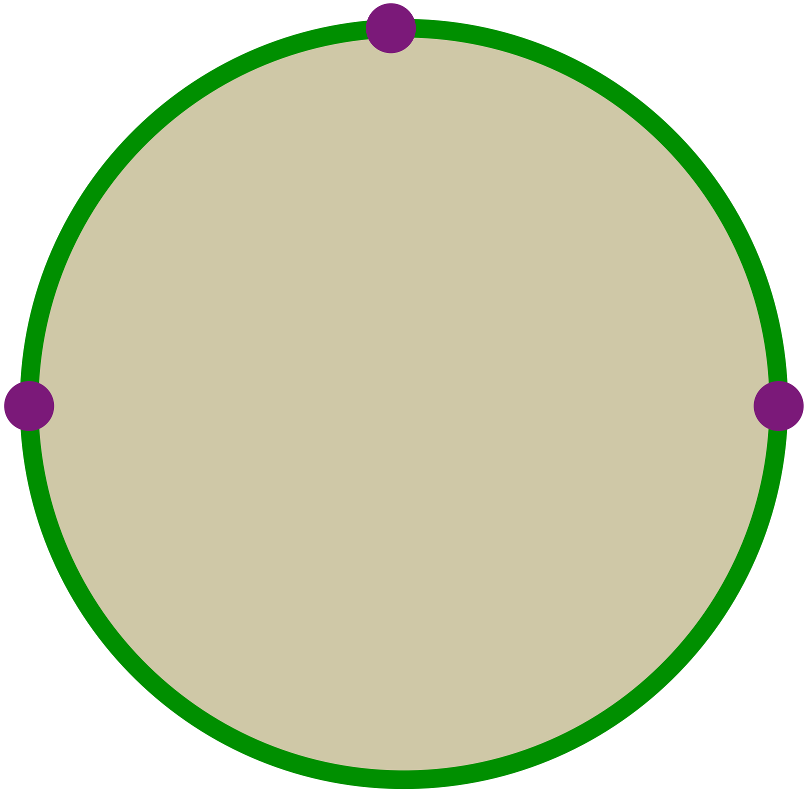

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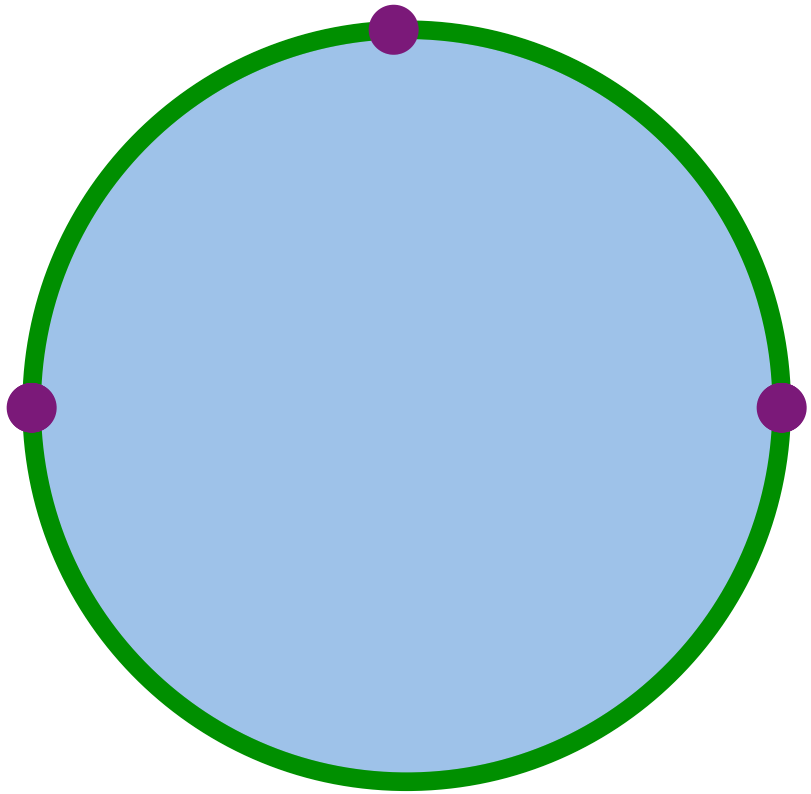

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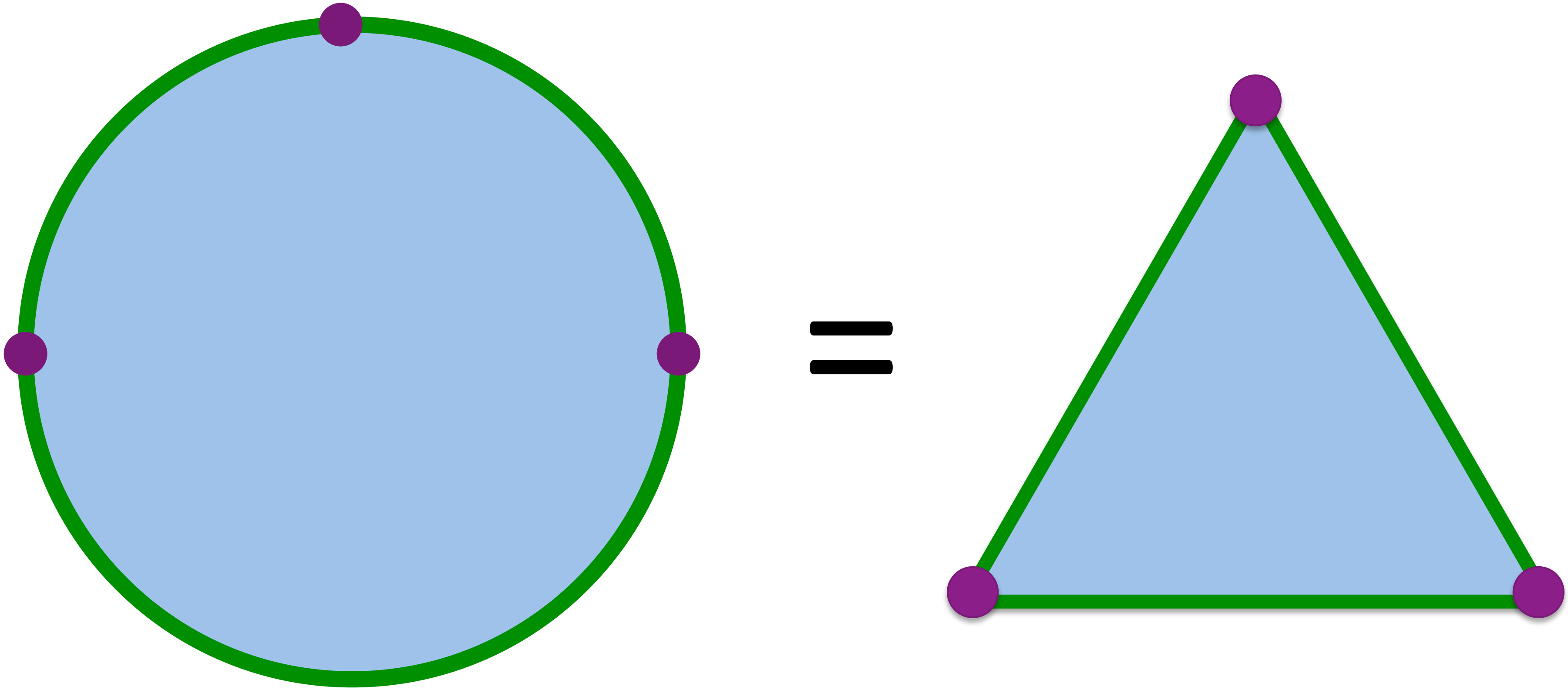

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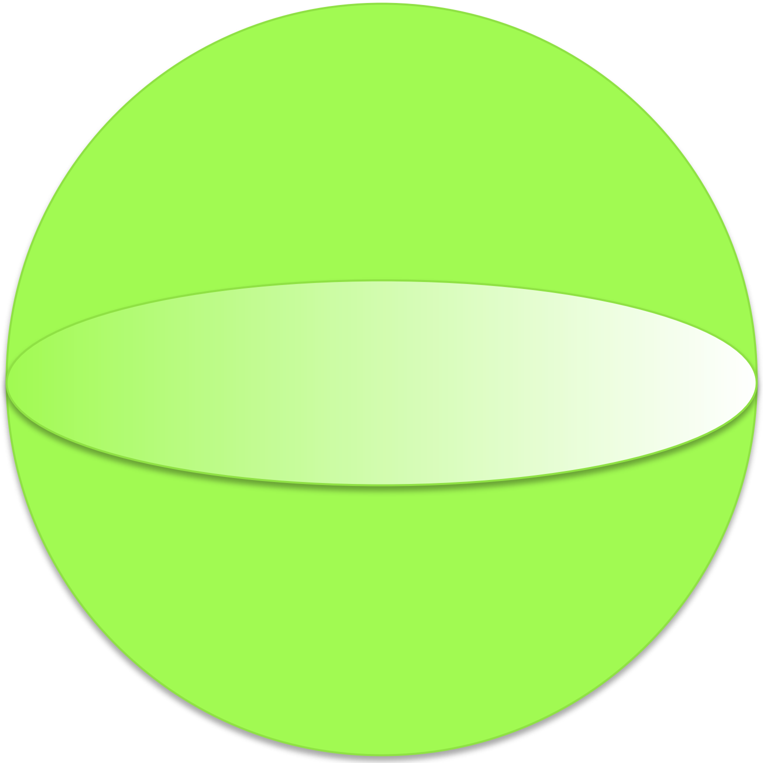

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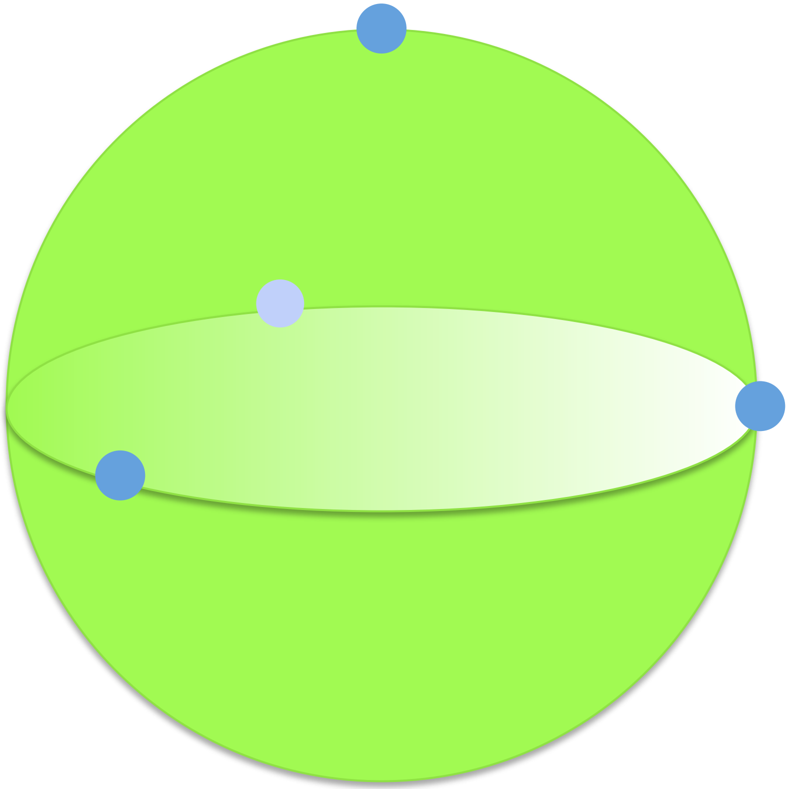

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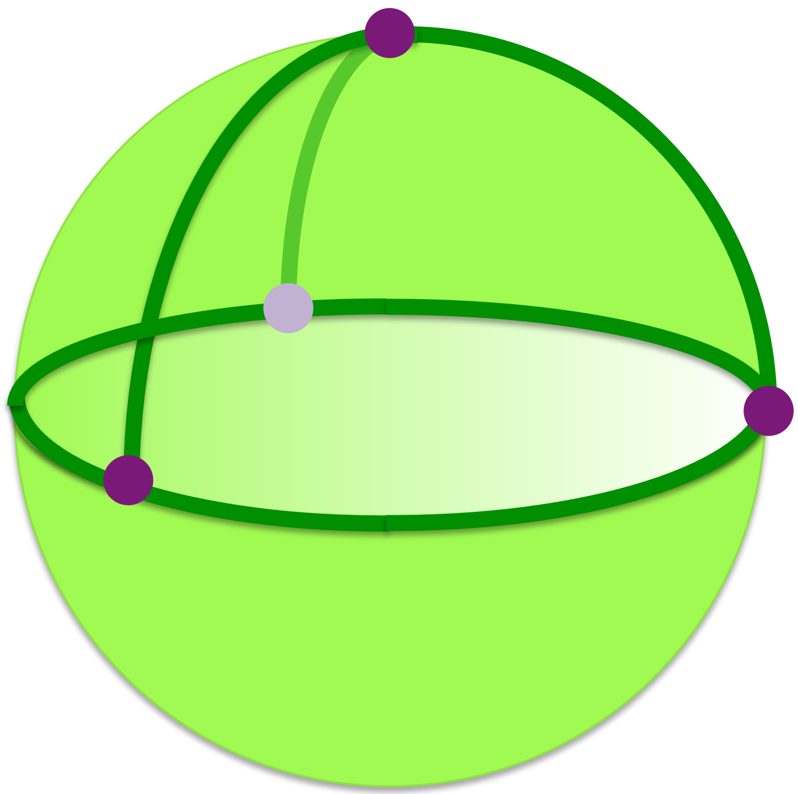

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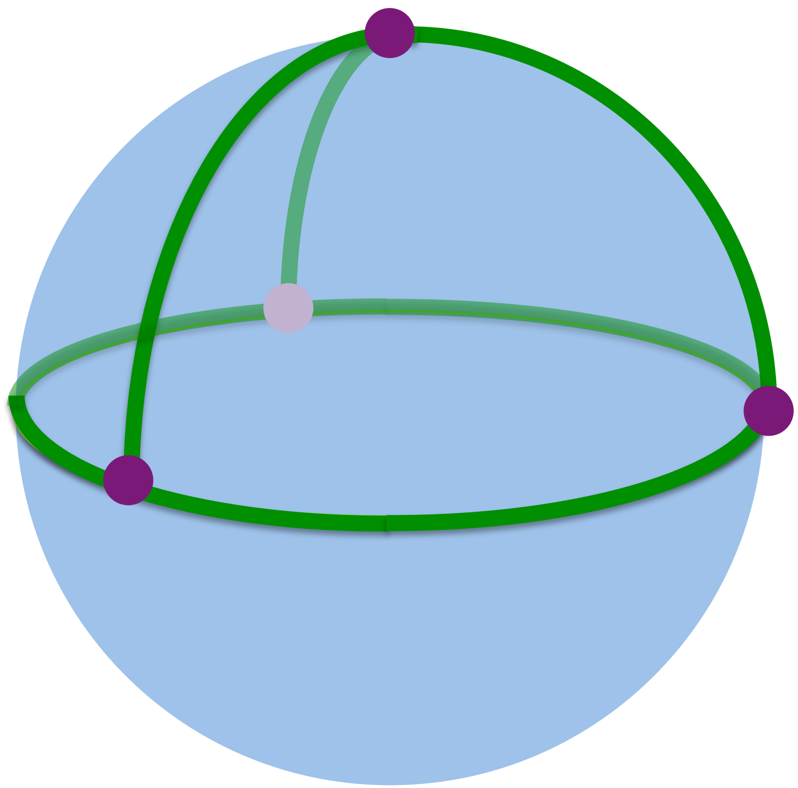

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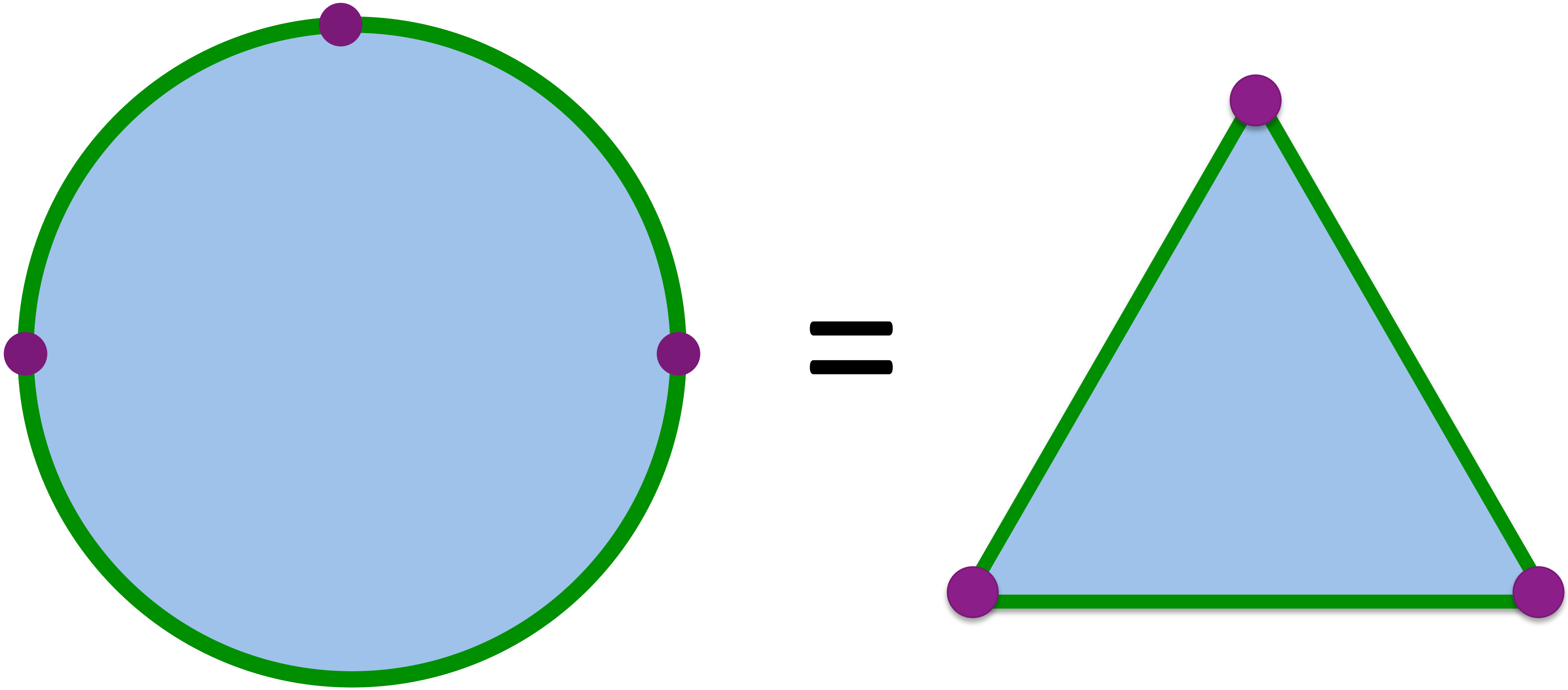

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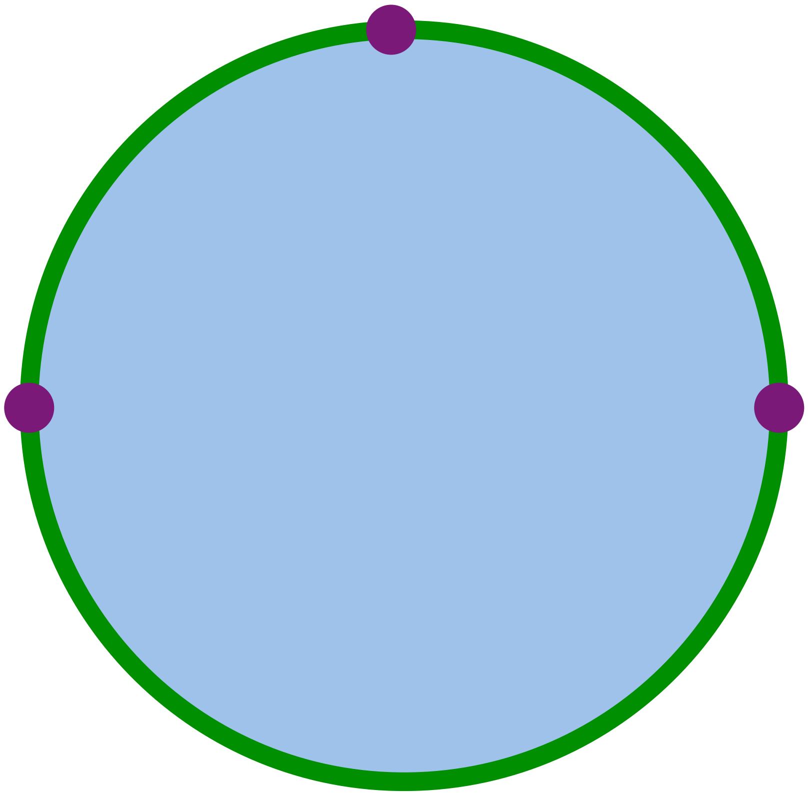

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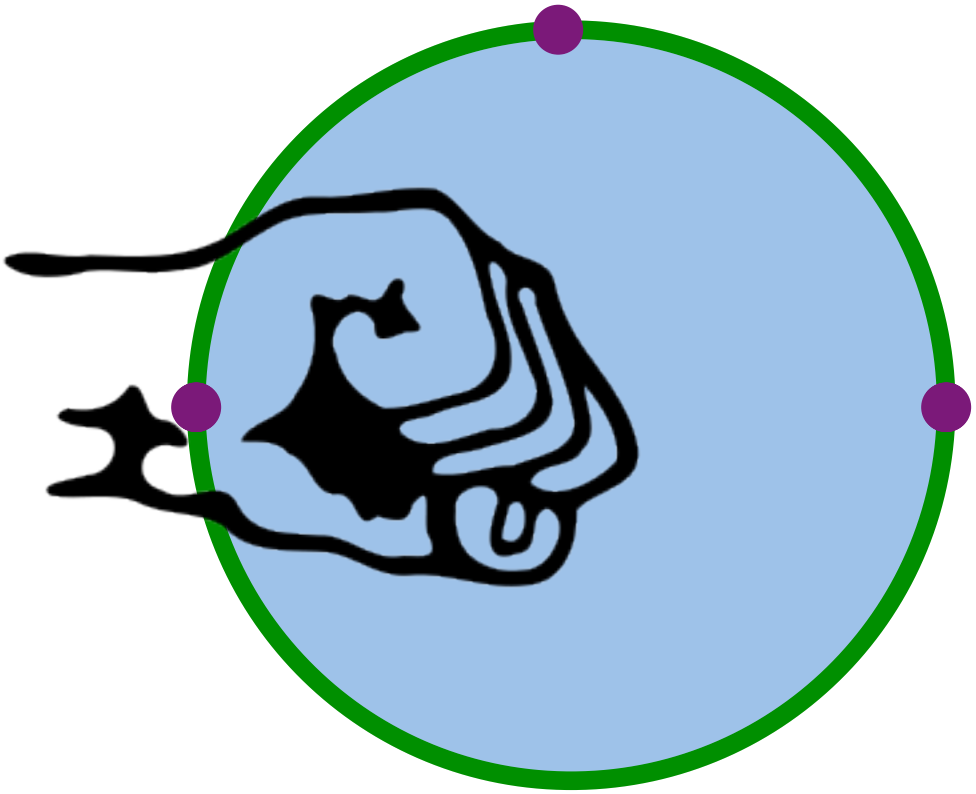

Fist image from http://openclipart.org/detail/1000/a-raised-fist-by-liftarn

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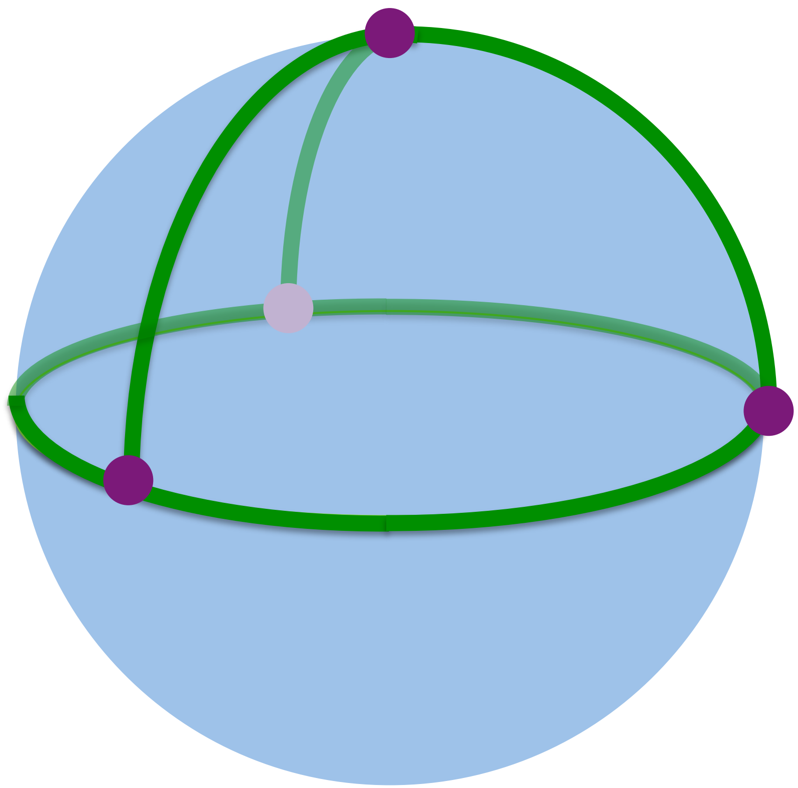

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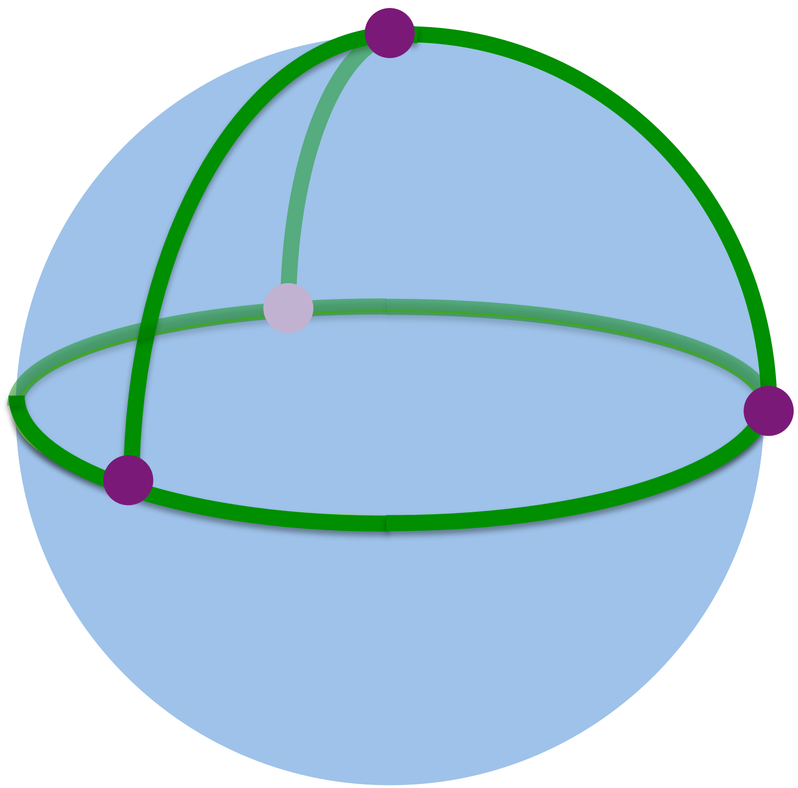

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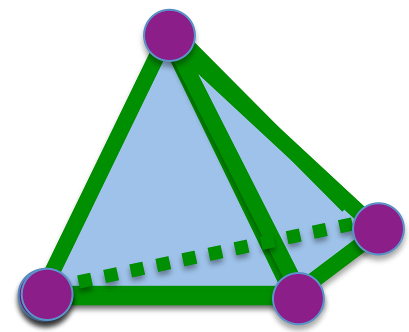


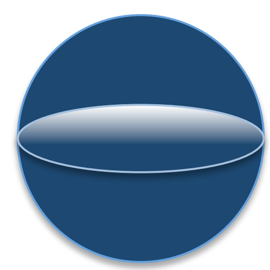
Creating a cell complex

Building block: n-cells = $\{ x \text{ in } R^n : || x || \leq 1 \}$

Examples: 0-cell = $\{ x \text{ in } R^0 : ||x|| < 1 \}$ 

1-cell =open interval =$\{ x \text{ in } R : ||x|| < 1 \}$ 

2-cell = open disk = $\{ x \text{ in } R^2 : ||x|| < 1 \}$ 

3-cell = open ball = $\{ x \text{ in } R^3 : ||x|| < 1 \}$ 

Building blocks for a simplicial complex

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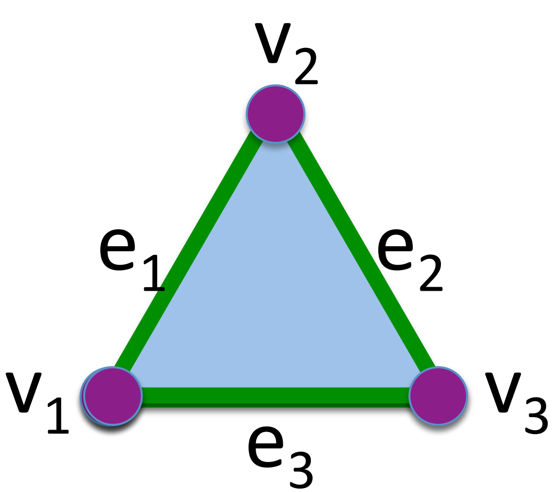

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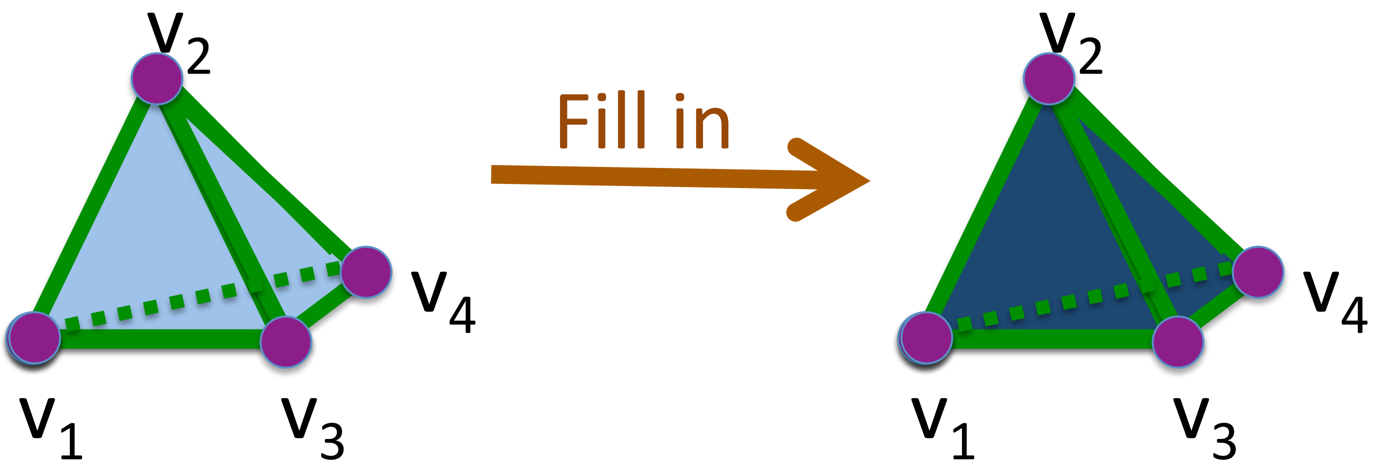

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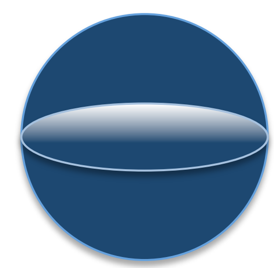
3-cell = open ball = $\{ x \text{ in } R^3 : ||x|| < 1 \}$ 

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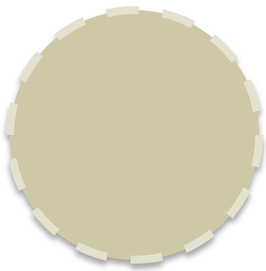
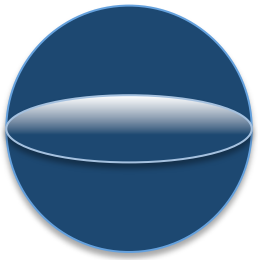
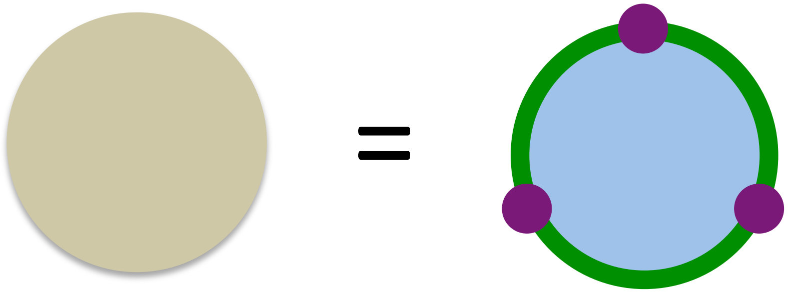
2-cell = open disk = $\{ x \text{ in } R^2 : ||x|| < 1 \}$ 

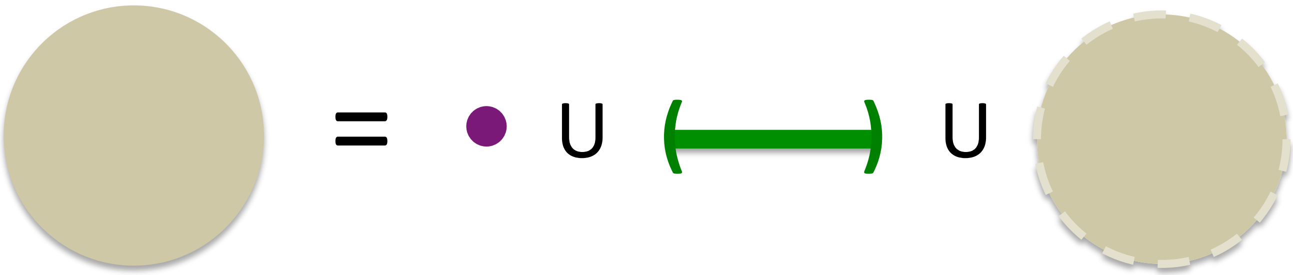
3-cell = open ball = $\{ x \text{ in } R^3 : ||x|| < 1 \}$ 

Example: disk = { x in R^2 : ||x|| ≤ 1 }

Simplicial complex



Cell complex



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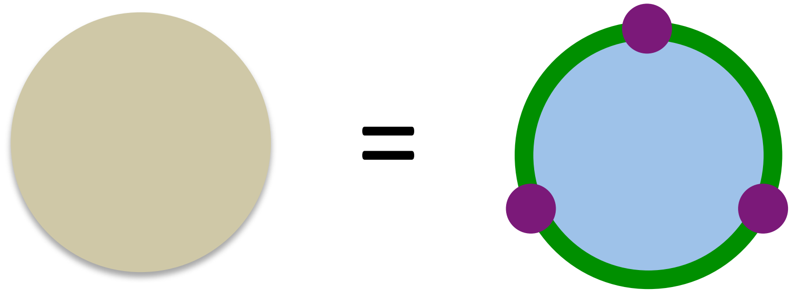

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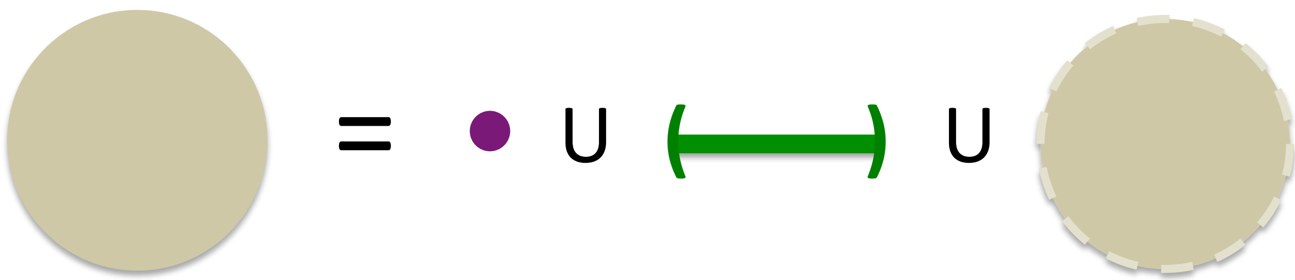

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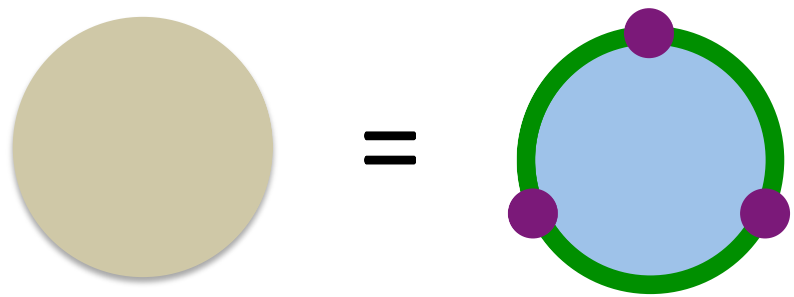
Simplicial complex



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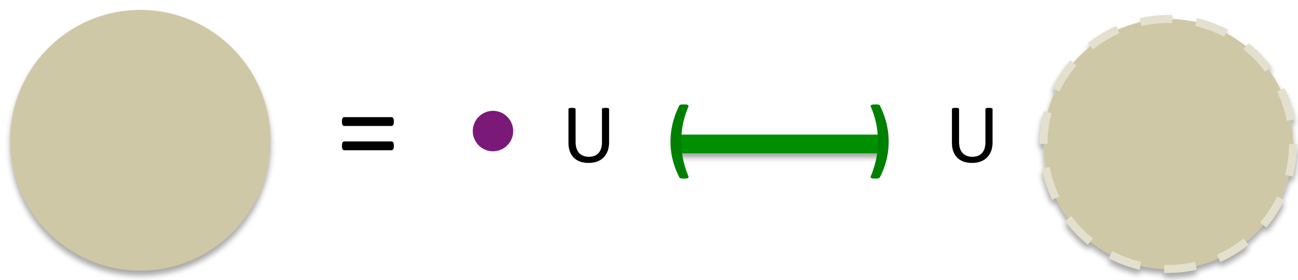

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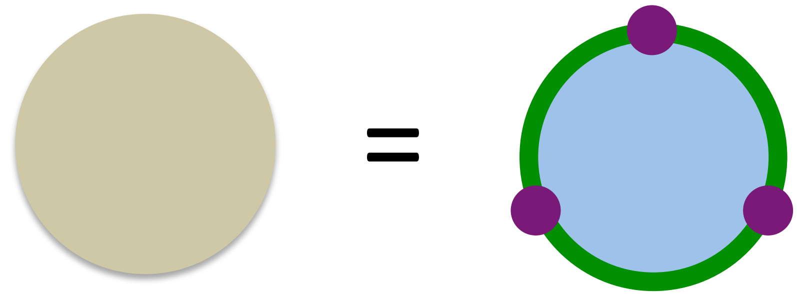

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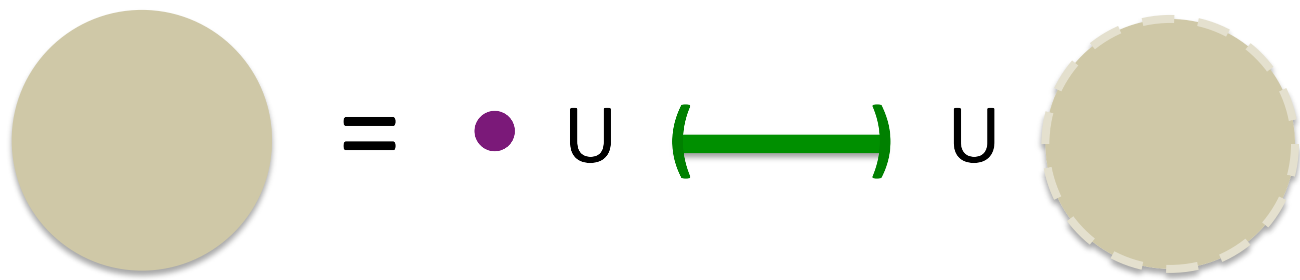

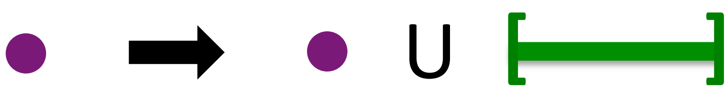

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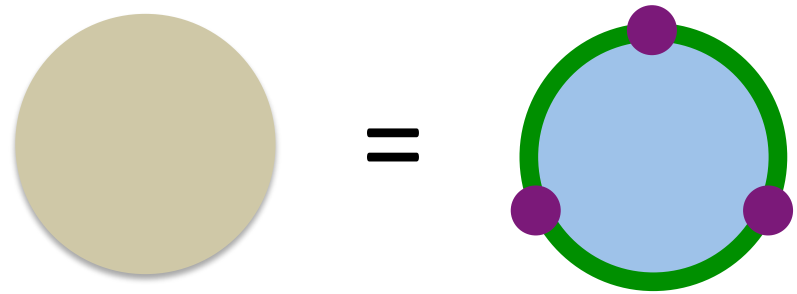

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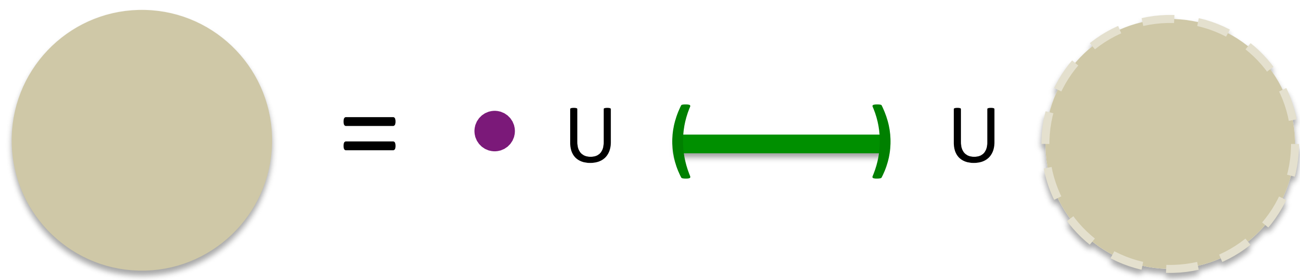

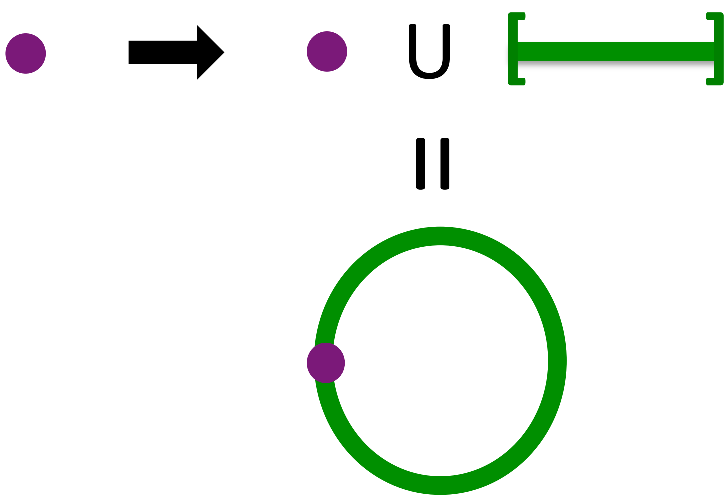

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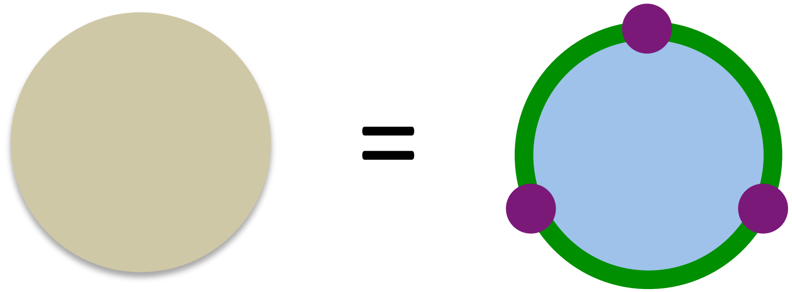
Simplicial complex



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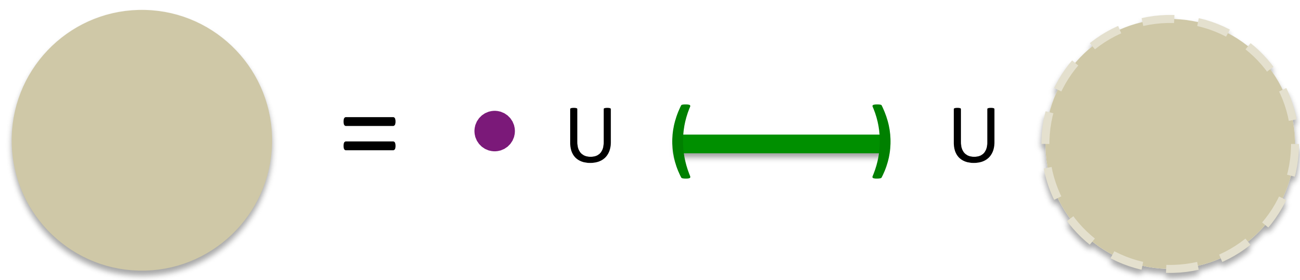
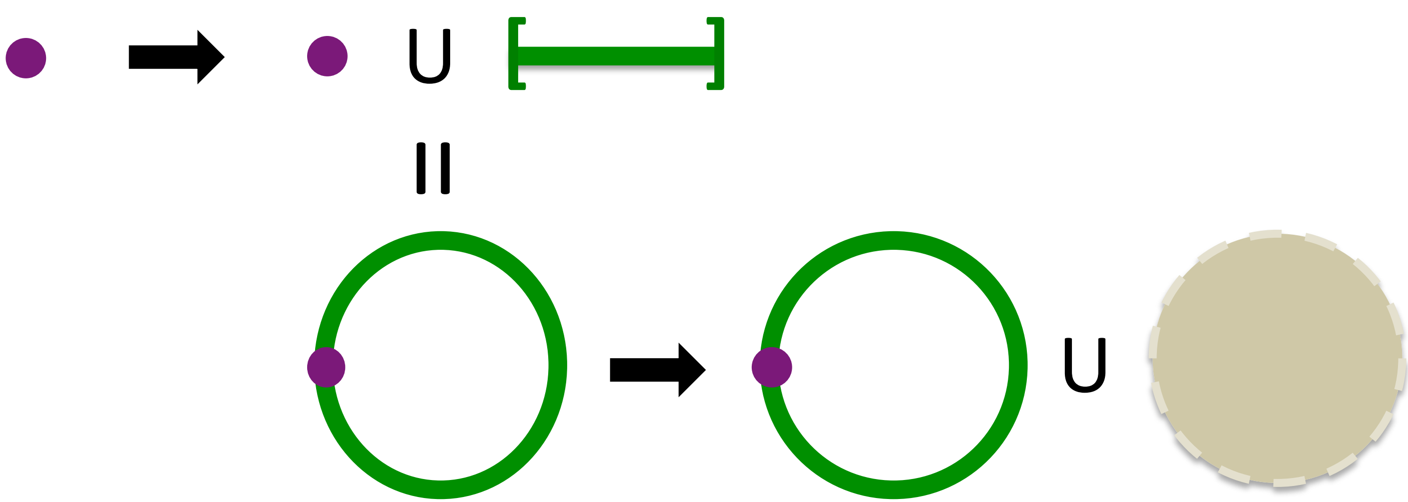
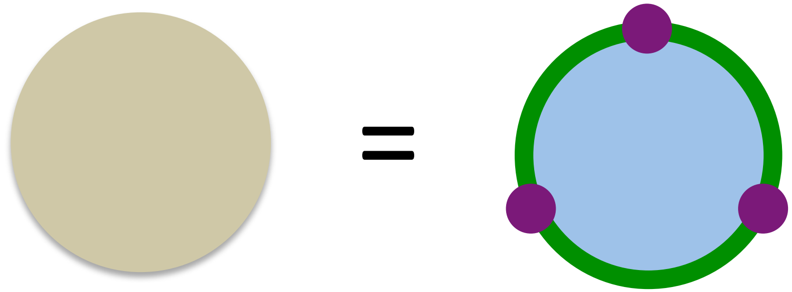


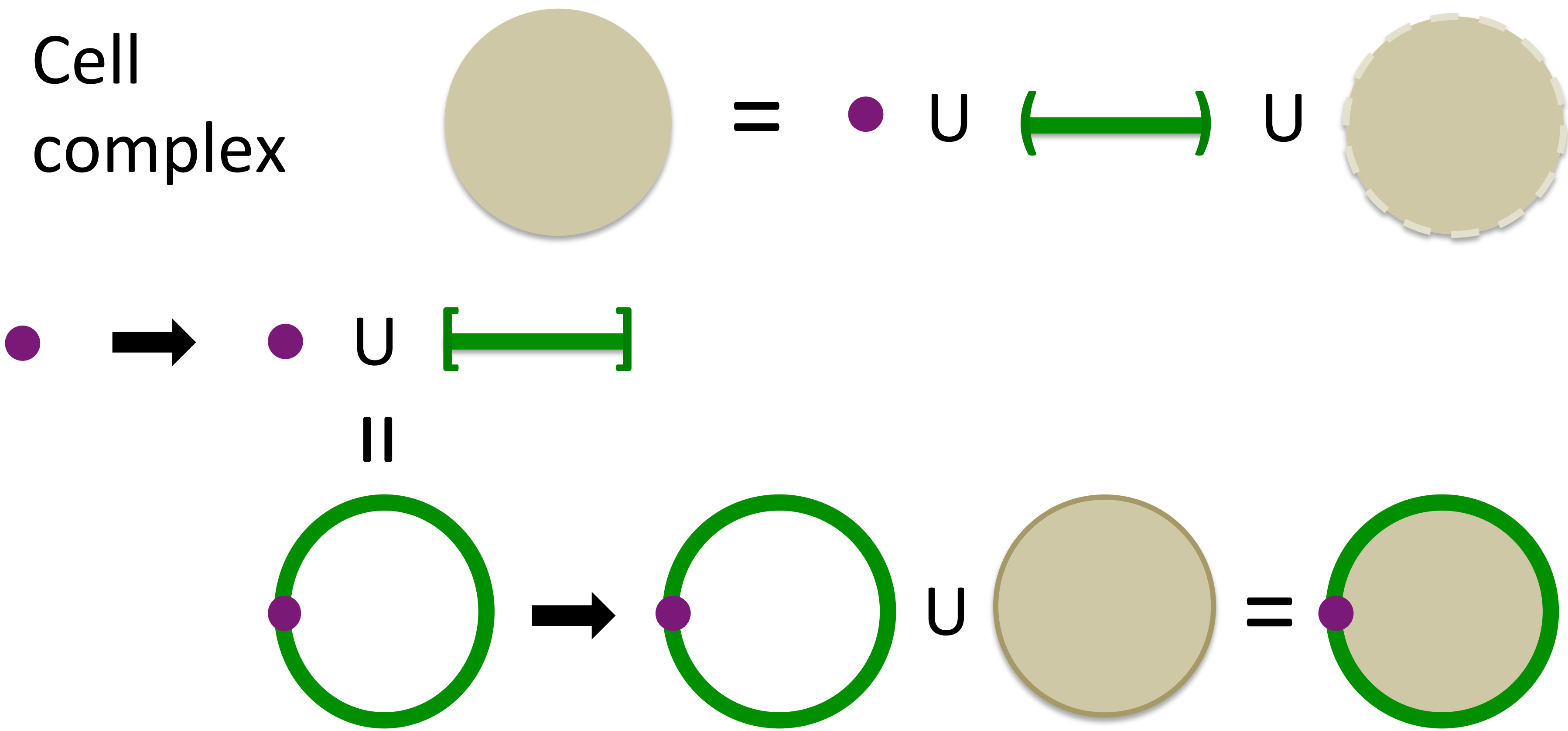


Example: disk = { x in R^2 : ||x|| ≤ 1 }

Simplicial complex



Cell complex



Euler characteristic (simple form):

χ = number of vertices – number of edges + number of faces

Or in short-hand,

$$\chi = |V| - |E| + |F|$$

where V = set of vertices

E = set of edges

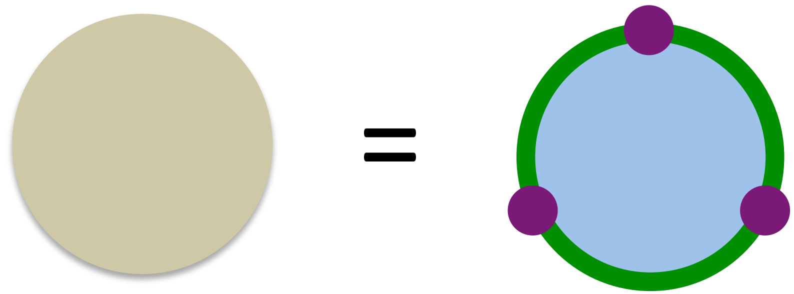
F = set of 2-dimensional faces

& the notation $|X|$ = the number of elements in the set X.

Example: disk = $\{ x \text{ in } R^2 : \|x\| \leq 1 \}$

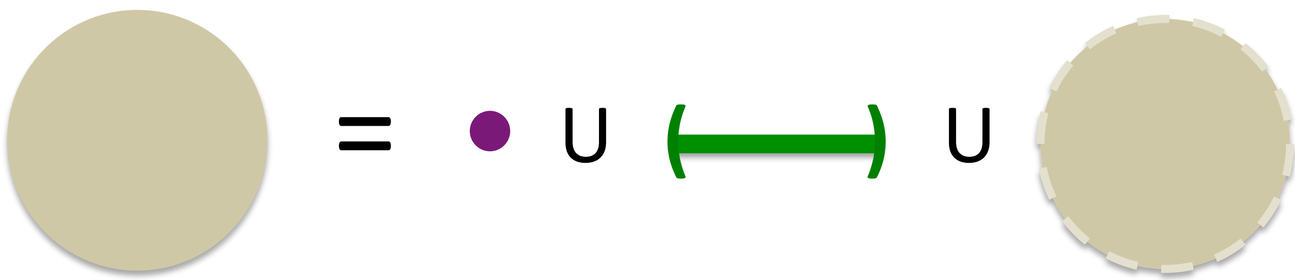
Simplicial complex

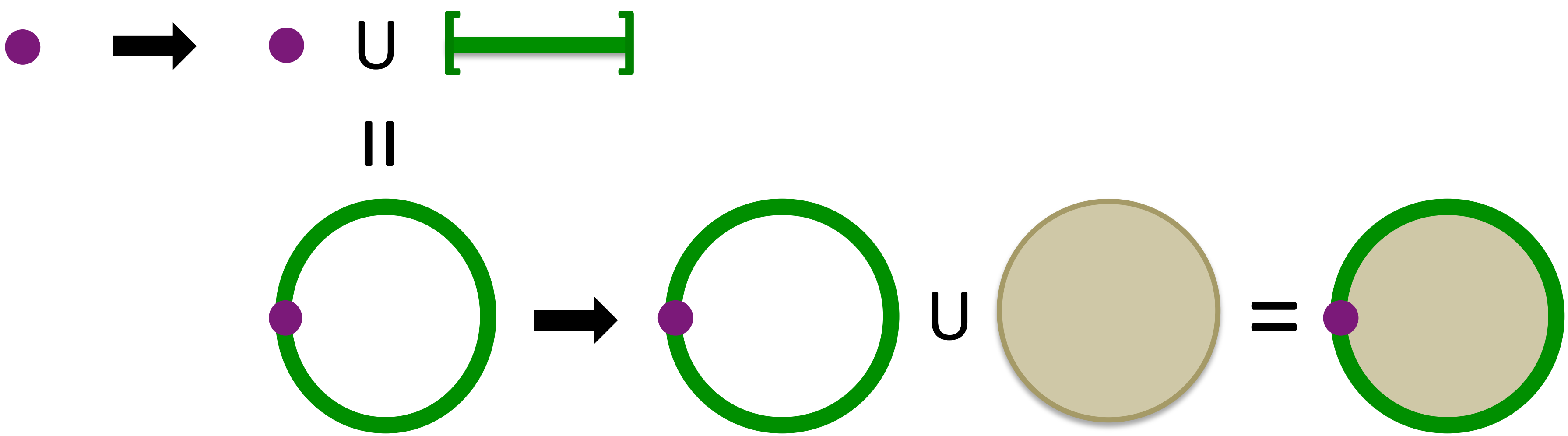
3 vertices, 3 edges, 1 triangle



Cell complex

1 vertex, 1 edge,
1 disk.





Euler characteristic:

Given a simplicial complex $\mathcal{C}$,

let C_n = the set of n-dimensional simplices in $\mathcal{C}$, and

let $|C_n|$ denote the number of elements in C_n. Then

$$\chi = |C_0| - |C_1| + |C_2| - |C_3| + \ldots$$

$$= \Sigma (-1)^n |C_n|$$

Euler characteristic:

Given a cell complex $\mathcal{C}$,

let C_n = the set of n-dimensional cells in $\mathcal{C}$, and

let $|C_n|$ denote the number of elements in C_n. Then

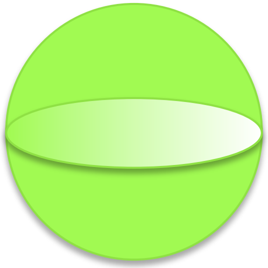
$$\chi = |C_0| - |C_1| + |C_2| - |C_3| + \ldots$$

$$= \Sigma (-1)^n |C_n|$$

Example: sphere = { x in R^3 : ||x|| = 1 }

Simplicial complex

4 vertices, 6 edges, 4 triangles



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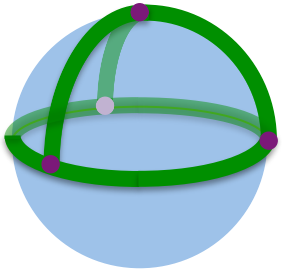
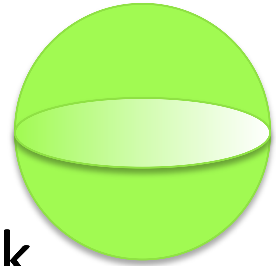


Cell Complex

1 vertex, 1 disk



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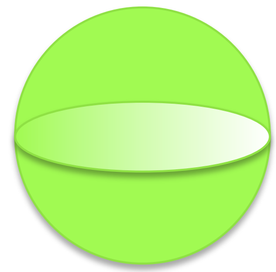
Example: sphere = { x in R^3 : $||x|| = 1$ }

Simplicial complex



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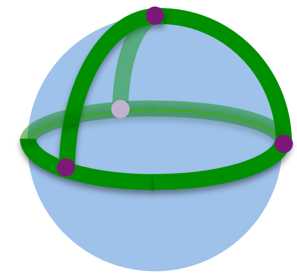
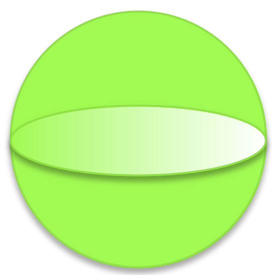


Cell complex



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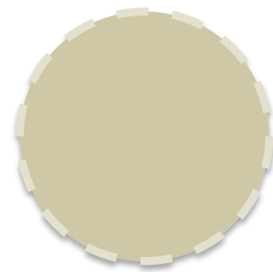
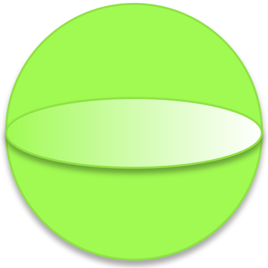


Example: sphere = { x in R^3 : ||x|| = 1 }

Simplicial complex



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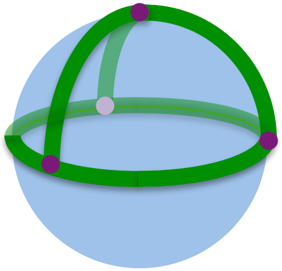
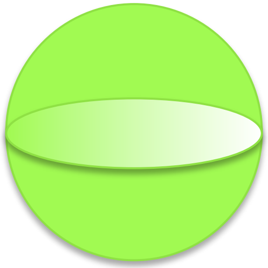


Cell complex



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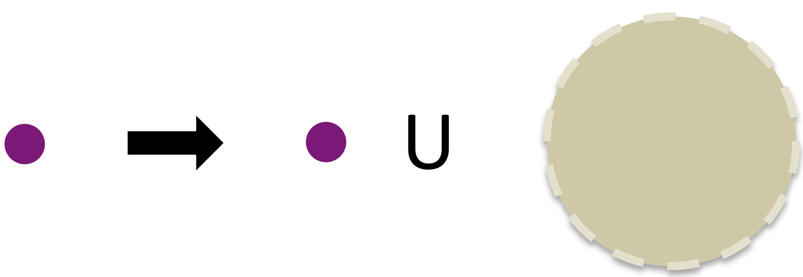
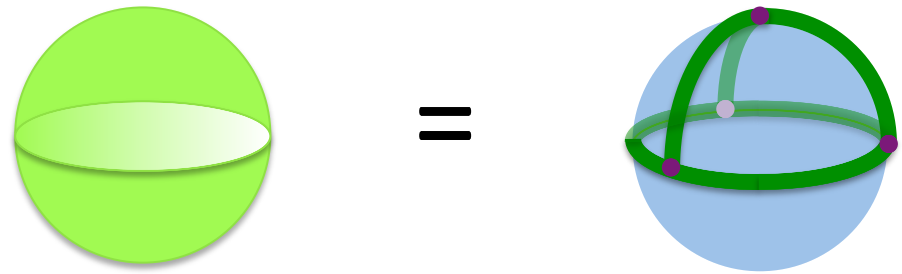
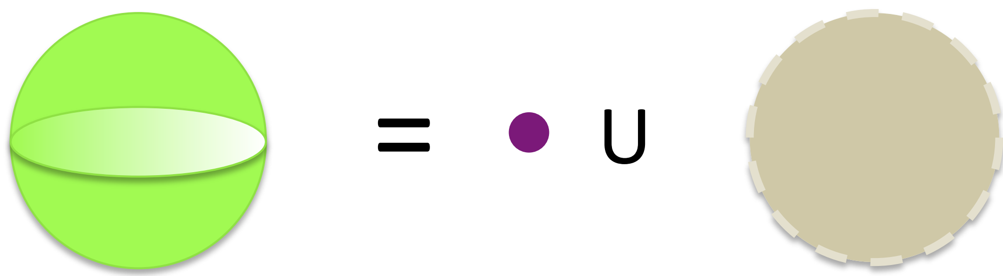


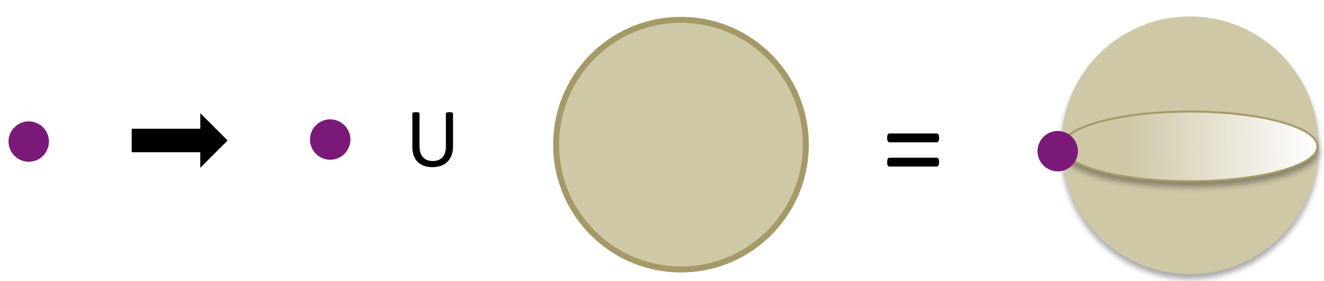
Example: sphere = { x in R^3 : | |x | | = 1 }

Simplicial complex



Cell complex





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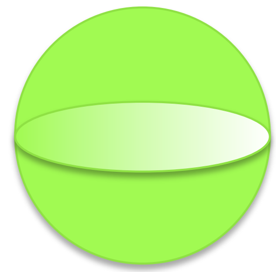
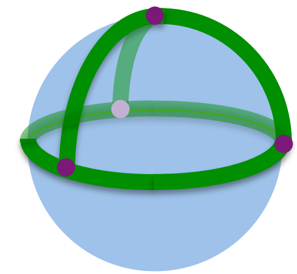
Simplicial complex

 = 

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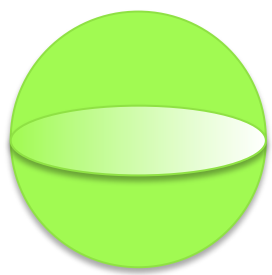
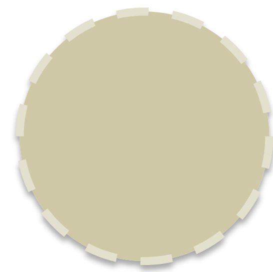
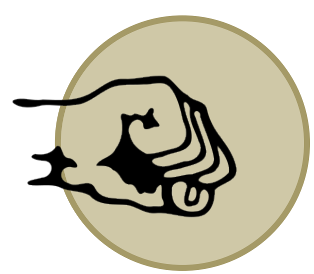
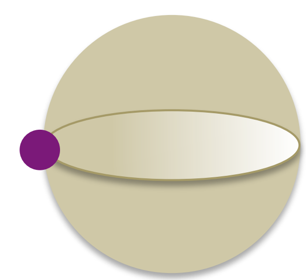
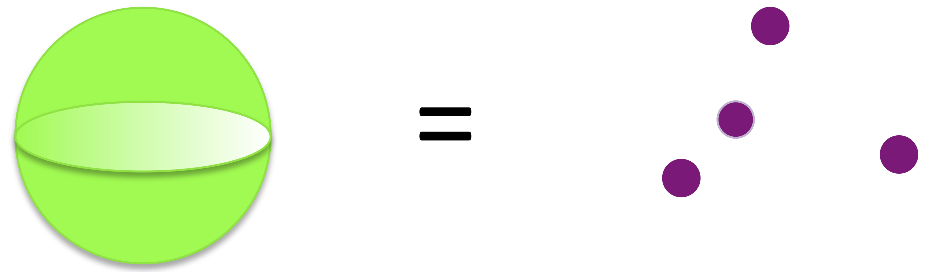
 = • U 

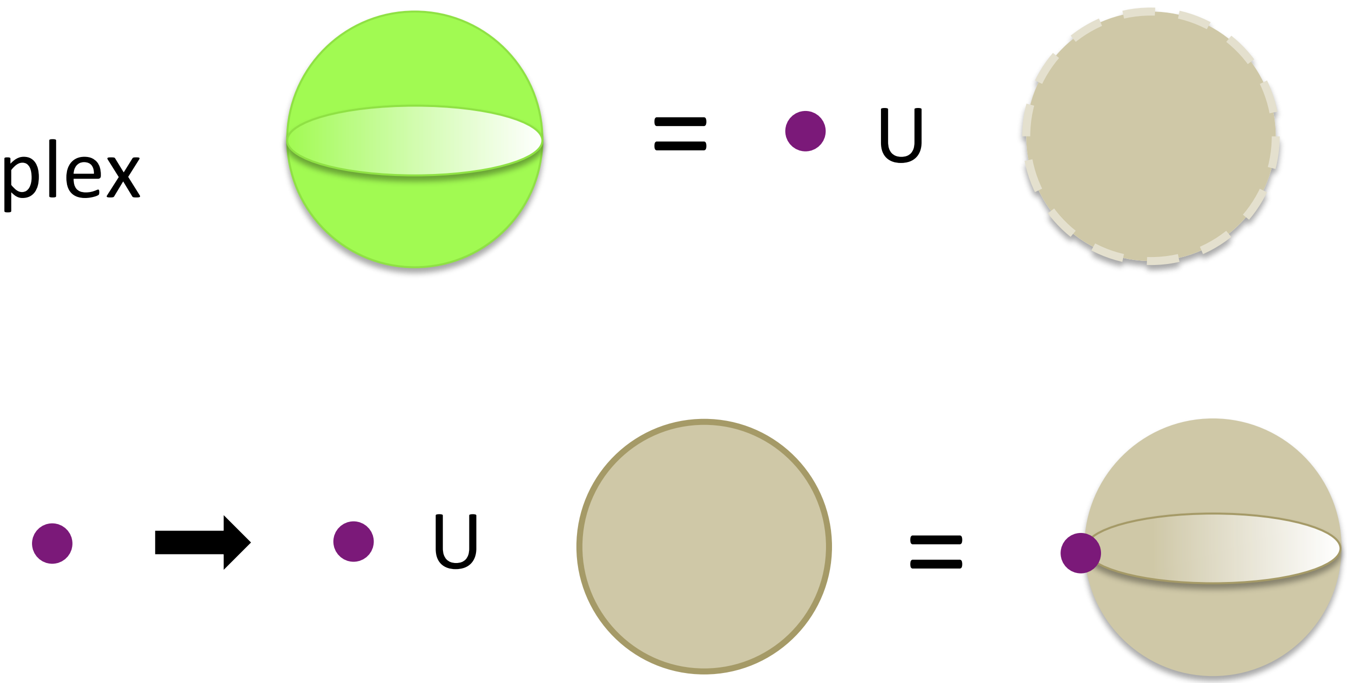
• ➡ • U  = 

Fist image from http://openclipart.org/detail/1000/a-raised-fist-by-liftarn

Example: sphere = { x in R^3 : ||x|| = 1 }



Cell complex



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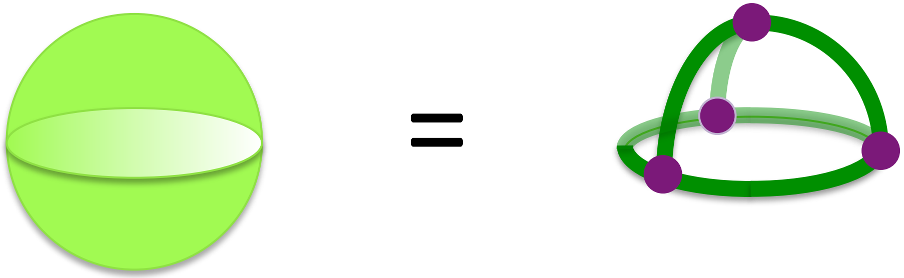
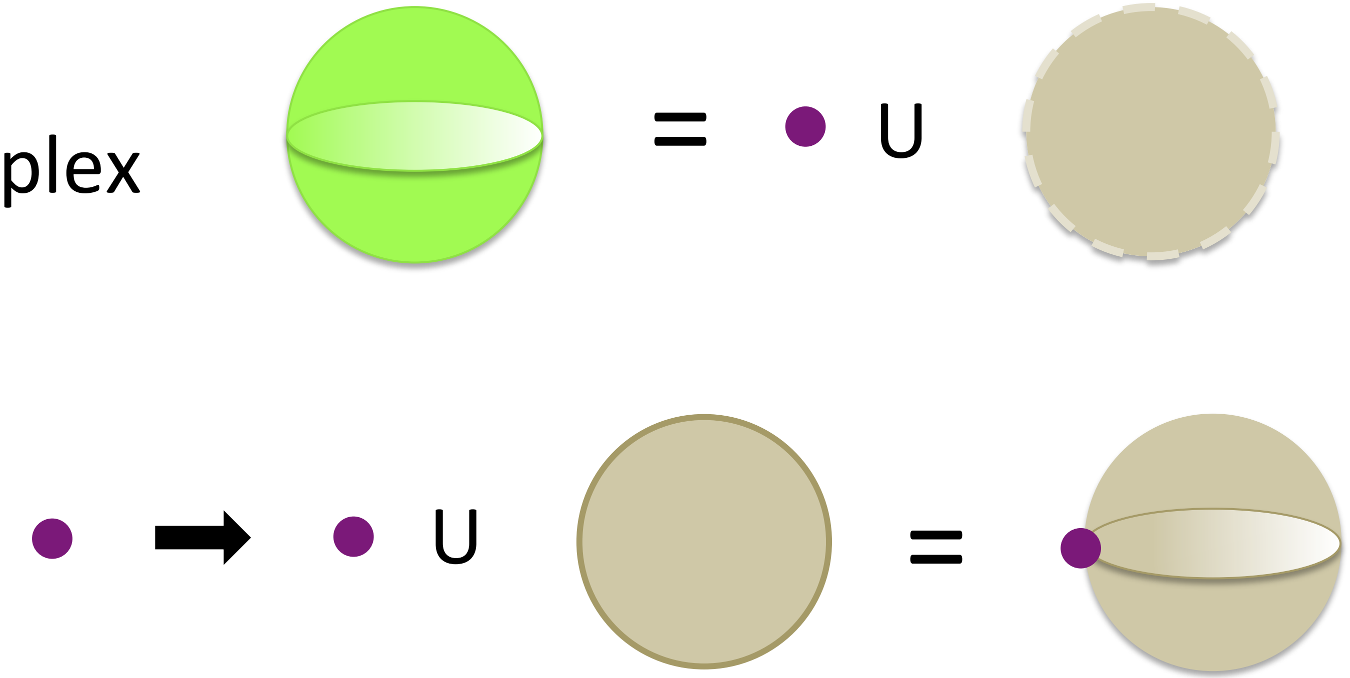
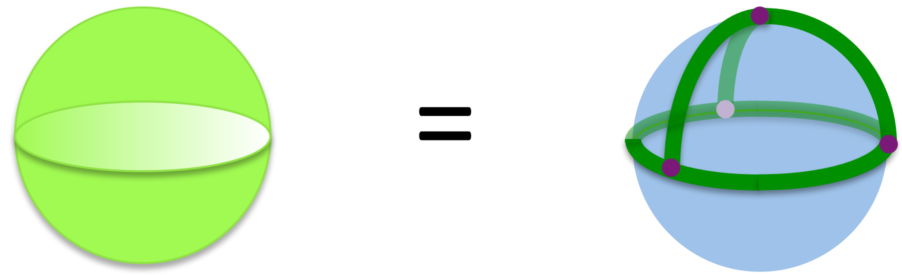


Cell complex



Example: sphere = { x in R^3 : $||x||$ = 1 }

Simplicial complex
Cell complex



Cell
complex