Lecture 4: Addition (and free vector spaces)

of a series of preparatory lectures for the Fall 2013 online course MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Target Audience: Anyone interested in **topological data analysis** including graduate students, faculty, industrial researchers in bioinformatics, biology, computer science, cosmology, engineering, imaging, mathematics, neurology, physics, statistics, etc.

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http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

A free abelian group generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i are integers.

Z = The set of integers = { …, -2, -1, 0, 1, 2, …}
= the set of all whole numbers (positive, negative, 0)

Addition:

$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

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Example: **Z**[, ]

4  + 2 

 – 2 

-3 

k  + n 

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$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

A free vector space over the field **F** generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i in **F**.

Examples of a field: **R** = set of real numbers

Q = set of rational numbers

$\mathbf{Z_2} = \{0, 1\}$

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where n_i in **F**.

Examples of a field:

R = set of real numbers:

$$\pi x + \sqrt{2}\, y + 3z \text{ is in } \mathbf{R}[x, y, z]$$

Q = set of rational numbers (i.e. fractions):

$$(\tfrac{1}{2})x + 4y \text{ is in } \mathbf{Q}[x, y]$$

$\mathbf{Z_2}$ = {0, 1}: $0x + 1y + 1w + 0z$ is in $\mathbf{Z_2}[x, y, z, w]$

Group	
Closure	x, y in G implies x + y is in G
Associative	(x + y) + z = x + (y + z)
Identity	0 + x = x = x + 0
Inverses	x + (-x) = 0 = (-x) + x

Examples of a group under addition:

R = set of real numbers

Q = set of rational numbers.

Z = set of integers.

$\mathbf{Z}_2$ = {0, 1}

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Abelian Group	
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Examples of an abelian group under addition:

R = set of real numbers

Q = set of rational numbers.

Z = set of integers.

$\mathbf{Z_2} = \{0, 1\}$

Group	
Closure	x, y in G implies x y is in G
Associative	(x y) z = x (y z)
Identity	1 x = x = 1x
Inverses	$x (x^{-1}) = 1 = (x^{-1}) x$

Examples of a group under multiplication:

R – {0} = set of real numbers not including zero.

Q – {0} = set of rational numbers not including zero.

$\mathbf{Z_2}$ – {0} = {1}

Group	
Closure	x, y in G implies x y is in G
Associative	$(x\ y)\ z = x\ (y\ z)$
Identity	$1\ x = x = 1x$
Inverses	$x\ (x^{-1}) = 1 = (x^{-1})\ x$

Note that **Z** – {0} is not a group under multiplication.

F is a *field* if

(1) **F** is an abelian group under addition

(2) **F** – {0} is an abelian group under multiplication

(3) multiplication distributes across addition.

Field	Addition	Multiplication
Closure	x, y in G ➔ x+ y in G	closure
Associative	$(x + y) + z = x + (y + z)$	$(x\,y)\,z = x\,(y\,z)$
Identity	$0 + x = x = x + 0$	$1\,x = x = 1x$
Inverses	$x + (-x) = 0 = (-x) + x$	$x\,(x^{-1}) = 1 = (x^{-1})\,x$
Commutative	$x + y = y + x$	$(x\,y)\,z = x\,(y\,z)$
Distributive	$x\,(y + z) = x\,y + x\,z$	

Examples of a field: **R** = set of real numbers

Q = set of rational numbers

$\mathbf{Z_2} = \{0, 1\}$

A free vector space over the field **F** generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

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Examples of a field: **R** = set of real numbers

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$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

A free vector space over the field $\mathbf{Z_2}$ generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i in $\mathbf{Z_2}$.

Example: $\mathbf{Z_2}[x_1, x_2] = \{0, \ x_1, \ x_2, \ x_1 + x_2\}$

$$4x_1 + 2x_2 = 0x_1 + 0x_2 = 0 \mod 2$$

$$1x_1 + 0x_2 = x_1 \mod 2$$

$$0x_1 + 1x_2 = x_2 \mod 2$$

$$kx_1 + nx_2 \mod 2$$

$\mathbf{Z_2}$ = The set of integers mod 2 = {0, 1}

Addition:

$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)\,x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

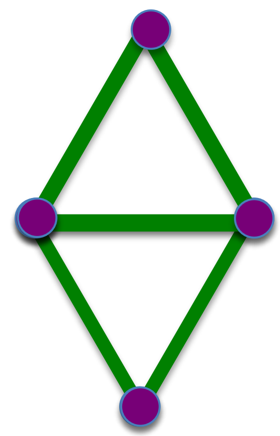
Example: $\mathbf{Z_2}[x_1, x_2] = \{0,\ x_1,\ x_2,\ x_1 + x_2\}$

$$1x_1 + 1x_1 = 2\,x_1 = 0 \bmod 2$$

$$(x_1 + x_2) + (x_1 + 0x_2) = 2\,x_1 + x_2 = x_2 \bmod 2$$

$$(1x_1 + 0x_2) + (0x_1 + 1x_2) = x_1 + x_2 \bmod 2$$

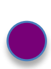
Example 2 from lecture 3:



4 vertices + 5 edges

4v + 5e

v = vertex



e = edge

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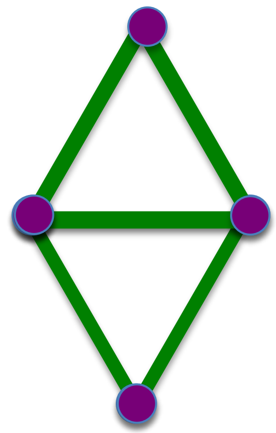


0 vertices + 1 edges   mod 2

0v + 1e = e   mod 2

v = vertex

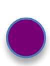
e = edge

Example 2 from lecture 3:

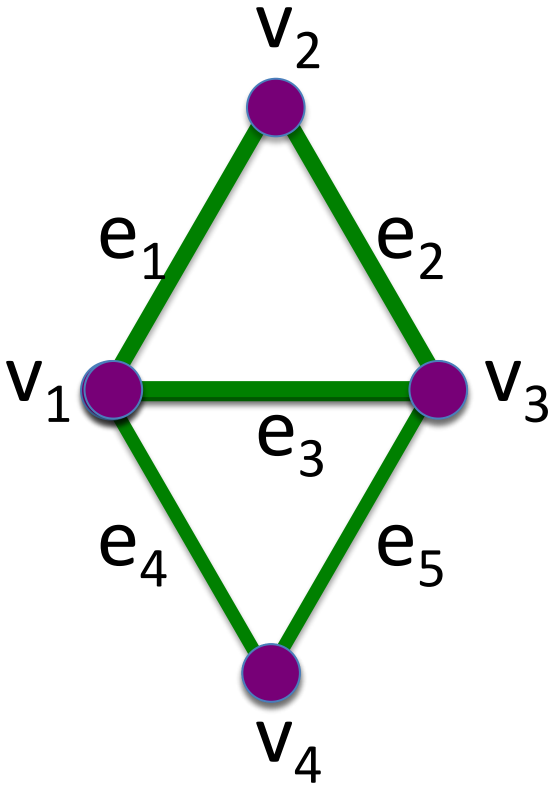
0 vertices + 1 edges   mod 2

0v + 1e = e   mod 2

v = vertex

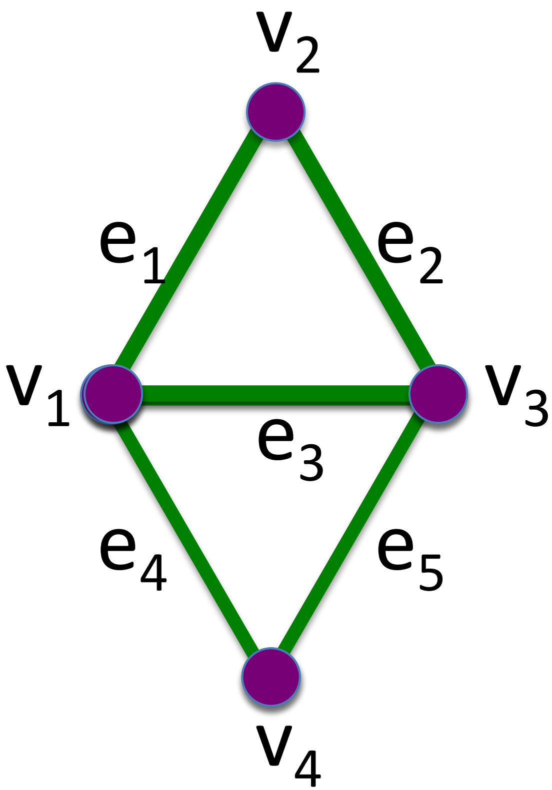


e = edge



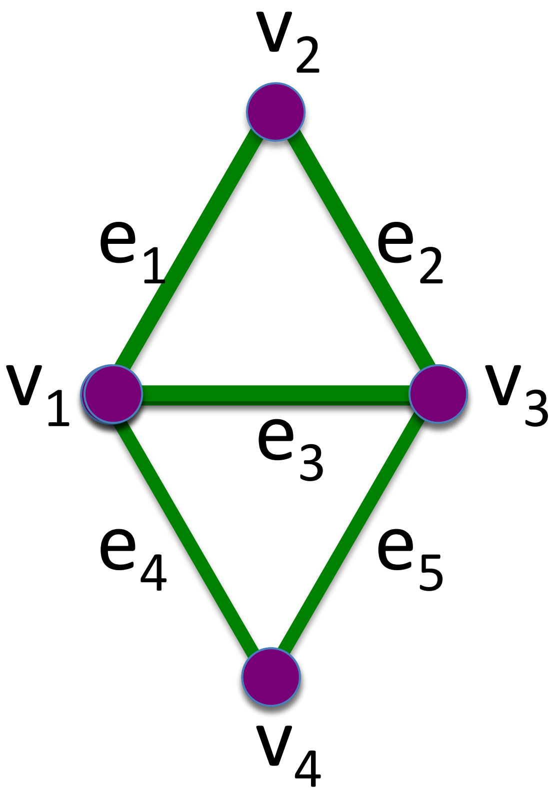
$v_1 + v_2 + v_3 + v_4 + e_1 + e_2 + e_3 + e_4 + e_5$

in $\mathbf{Z_2}[v_1, v_2, , v_3, v_4, e_1, e_2, , e_3, e_4, e_5]$



$v_1 + v_2 + v_3 + v_4$ in $\mathbf{Z}[v_1, v_{2,}, v_3, v_4]$

$e_1 + e_2 + e_3 + e_4 + e_5$ in $\mathbf{Z}[e_1, e_{2,}, e_3, e_4, e_5]$

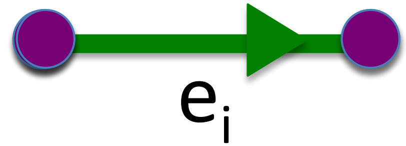


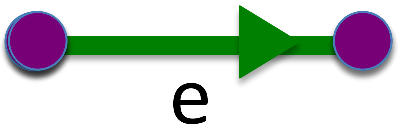
Note that $e_1 + e_2 + e_3$ is a cycle.

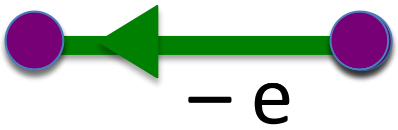
Note that $e_3 + e_5 + e_4$ is a cycle.

In **Z**[e_1, e_2, e_3, e_4, e_5]

Objects: oriented edges

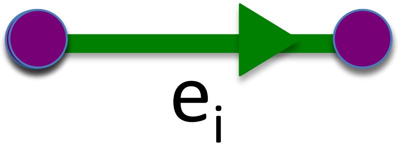


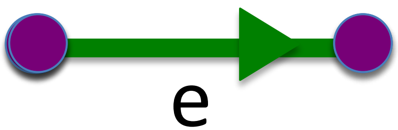


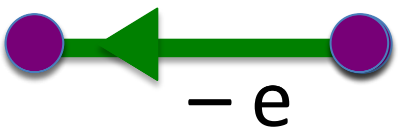


But in $\mathbf{Z_2}$, $1 = -1$. Thus $e = -e$

In $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$

Objects: oriented edges 





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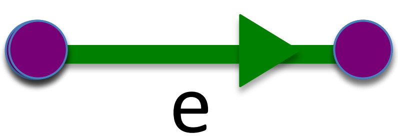

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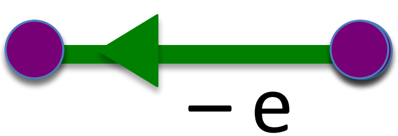
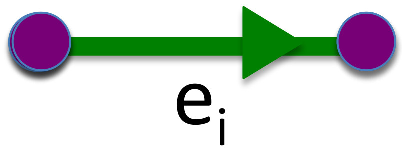


In $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$

Objects: oriented edges 

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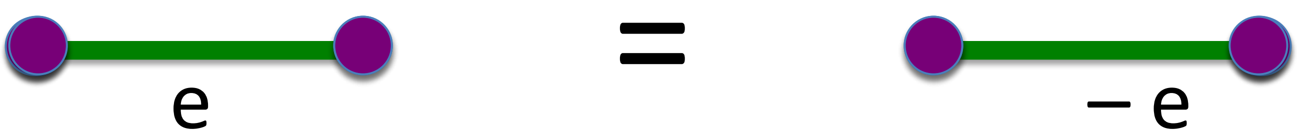
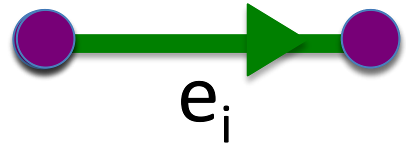
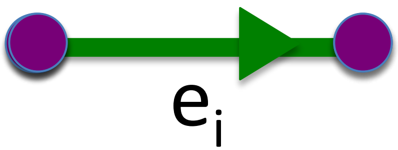
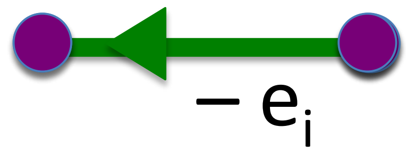


In $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$

Objects: oriented edges

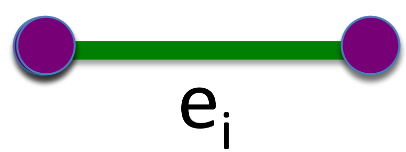




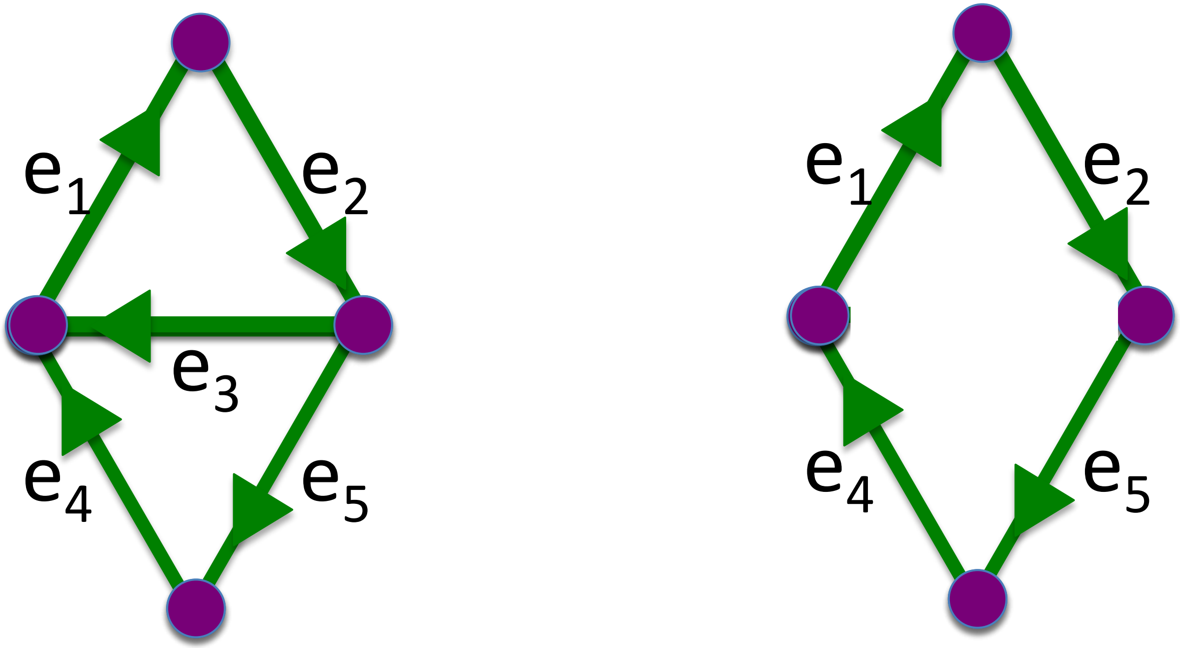


In $\mathbf{Z_2}[e_1, e_2, e_3, e_4, e_5]$

Objects: edges

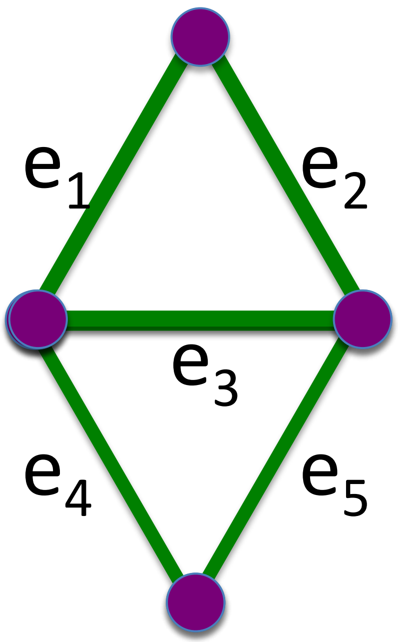


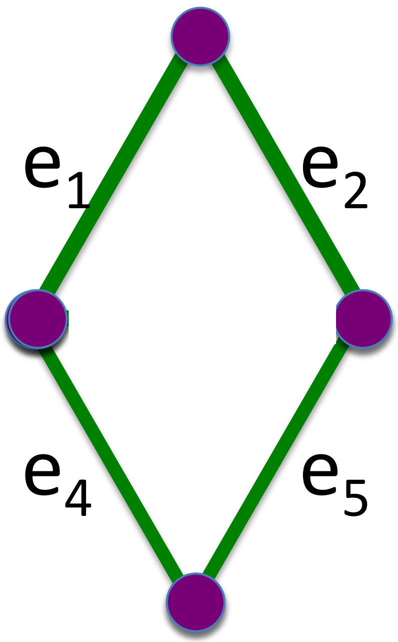
In $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$



$$(e_1 + e_2 + e_3) + (-e_3 + e_5 + e_4) = e_1 + e_2 + e_5 + e_4$$

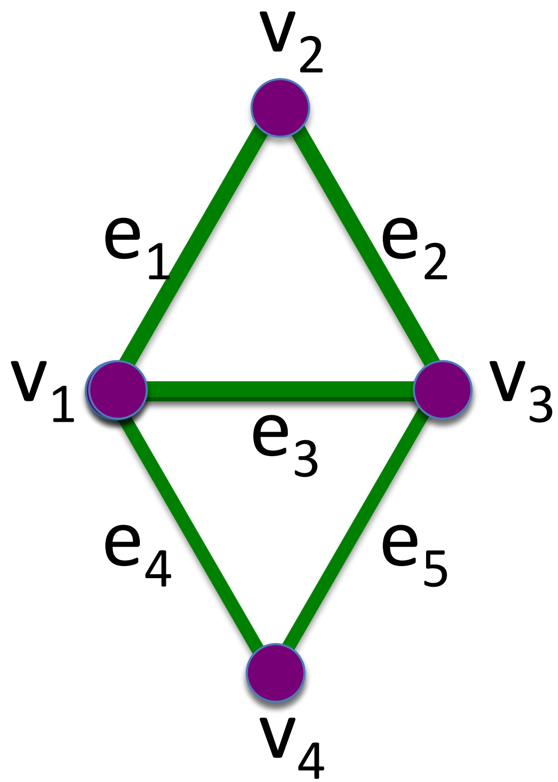
In $\mathbf{Z_2}[e_1, e_2, e_3, e_4, e_5]$

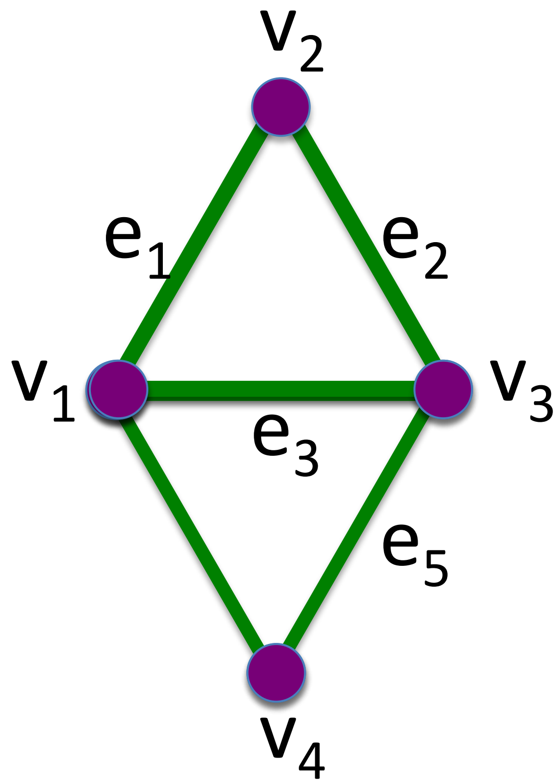




$$(e_1 + e_2 + e_3) + (e_3 + e_5 + e_4) = e_1 + e_2 + 2e_3 + e_5 + e_4$$

$$= e_1 + e_2 + e_5 + e_4$$

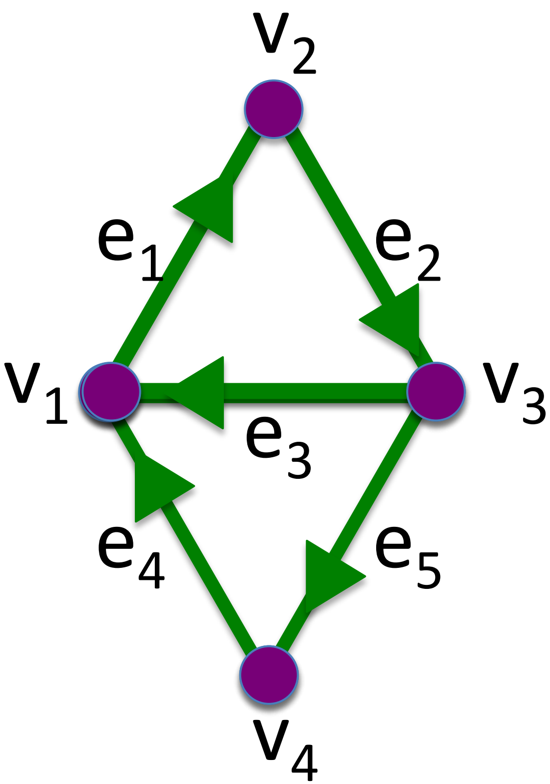




$$e_1 + e_2 + e_3 + e_1 + e_2 + e_5 + e_4 = 2e_1 + 2e_2 + e_3 + e_4 + e_5$$
$$= e_3 + e_4 + e_5$$

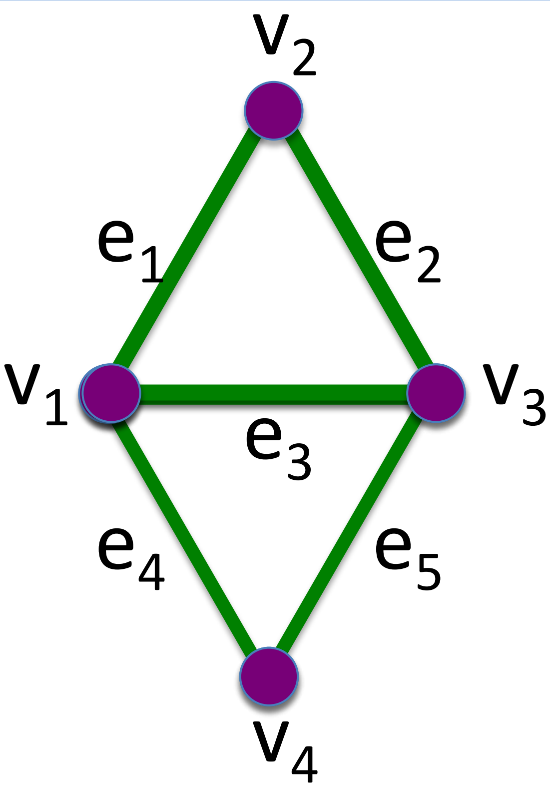
$$e_1 + e_2 + e_5 + e_4 + e_1 + e_2 + e_3 = e_3 + e_4 + e_5$$

In **Z**$[e_1, e_2, e_3, e_4, e_5]$



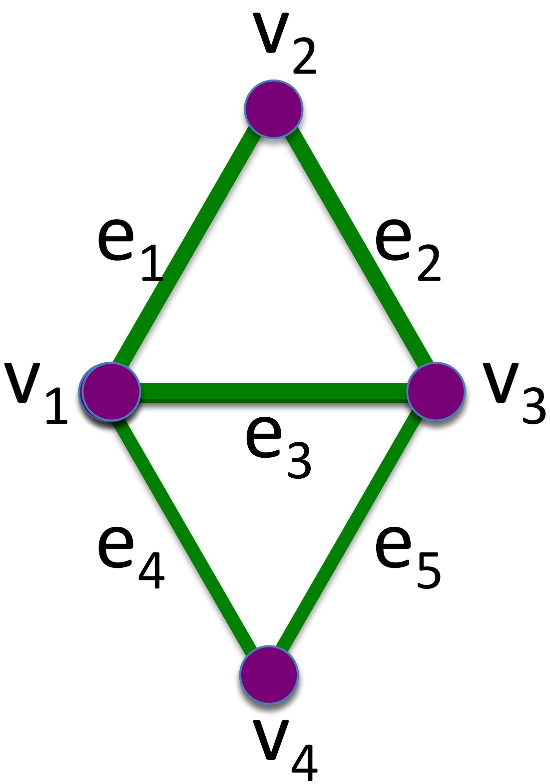
The boundary of $e_1 = v_2 - v_1$

In $\mathbf{Z_2}[e_1, e_2, e_3, e_4, e_5]$



The boundary of $e_1 = v_2 + v_1$

In $\mathbf{Z_2}[e_1, e_2, e_3, e_4, e_5]$



The boundary of $e_1 = v_2 + v_1$

The boundary of $e_2 = v_3 + v_2$

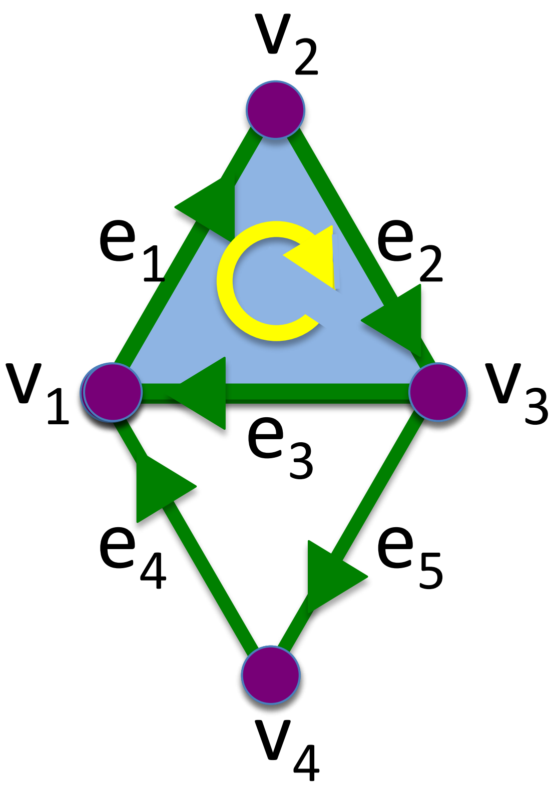
The boundary of $e_3 = v_1 + v_3$

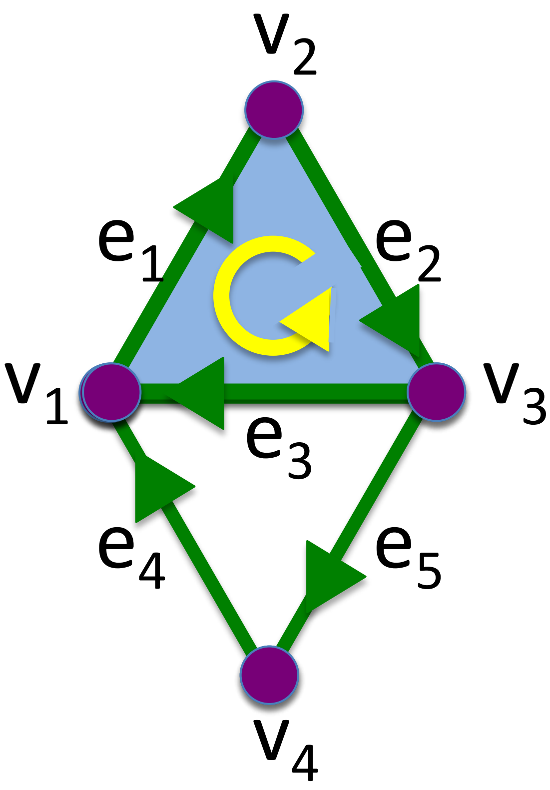
The boundary of $e_1 + e_2 + e_3$

$= v_2 + v_1 + v_3 + v_2 + v_1 + v_3 = 2v_1 + 2v_2 + 2v_3 = 0$

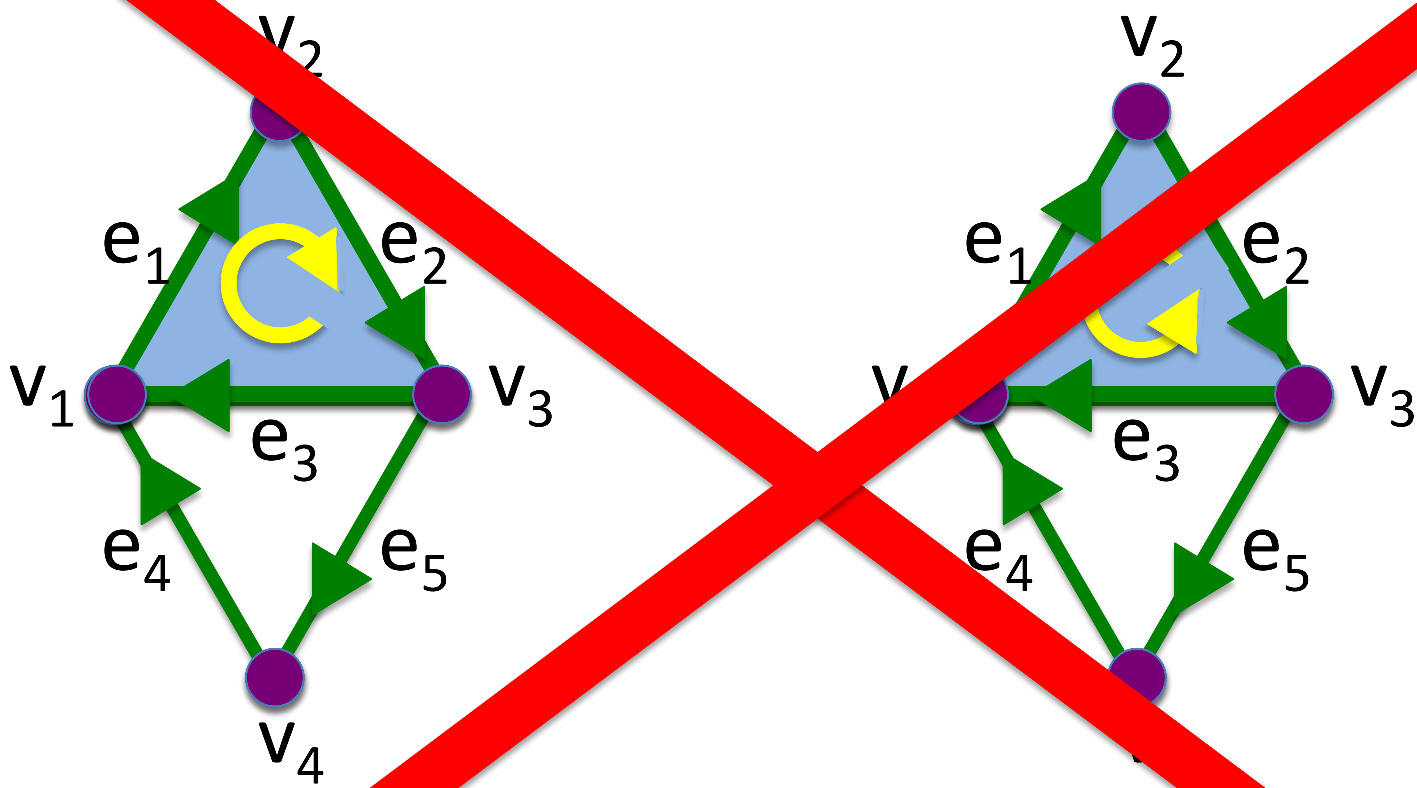
In **Z[f]**

Add an oriented face





Add an oriented face



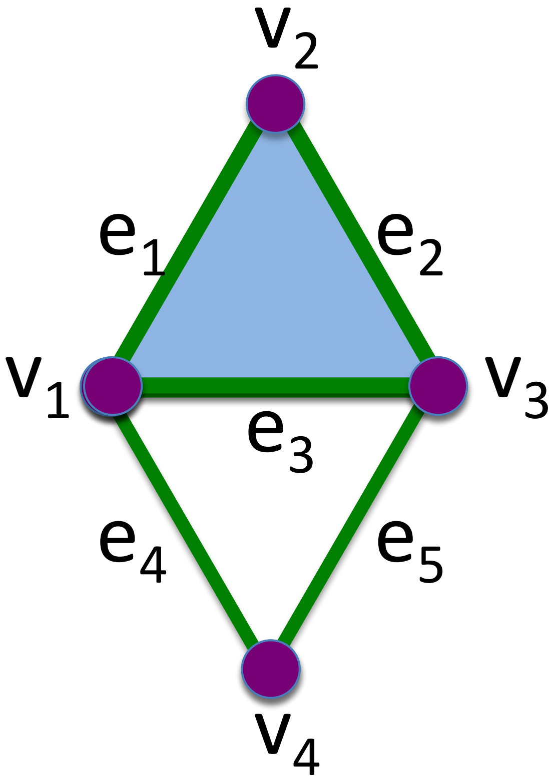
But with $\mathbf{Z_2}$

coefficients $+1 = -1$

so $f = -f$

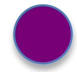
In $\mathbf{Z_2}[f]$

Add a face



Building blocks for a simplicial complex using $\mathbf{Z}_2$ coefficients

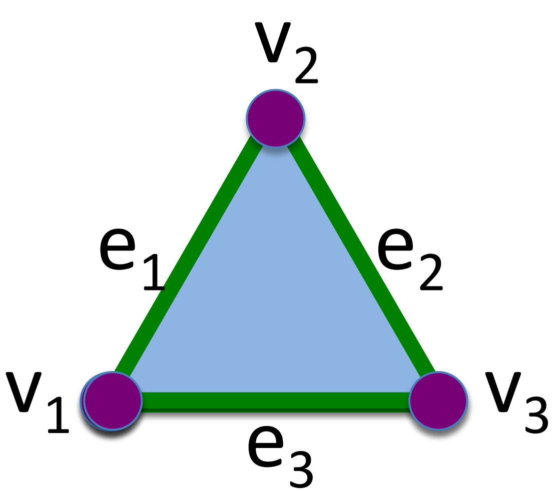
0-simplex = vertex = v



1-simplex = edge = $\{v_1, v_2\}$

Note that the boundary of this edge is $v_2 + v_1$

2-simplex = face = $\{v_1, v_2, v_3\}$



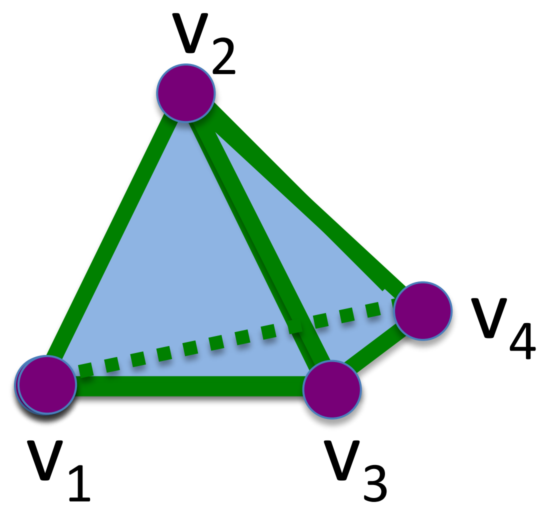
Note that the boundary of this face is the cycle

$$e_1 + e_2 + e_3$$

$$= \{v_1, v_2\} + \{v_2, v_3\} + \{v_1, v_3\}$$

Building blocks for a simplicial complex using $\mathbf{Z_2}$ coefficients

3-simplex = $\{v_1, v_2, v_3, v_4\}$ = tetrahedron



boundary of $\{v_1, v_2, v_3, v_4\}$ =
$\{v_1, v_2, v_3\} + \{v_1, v_2, v_4\} + \{v_1, v_3, v_4\} + \{v_2, v_3, v_4\}$

---

n-simplex = $\{v_1, v_2, \ldots, v_{n+1}\}$