Lecture 2: Addition (and free abelian groups)

of a series of preparatory lectures for the Fall 2013 online course MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Target Audience: Anyone interested in **topological data analysis** including graduate students, faculty, industrial researchers in bioinformatics, biology, computer science, cosmology, engineering, imaging, mathematics, neurology, physics, statistics, etc.

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http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

A free abelian group generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i are integers.

Z = The set of integers = { ..., -2, -1, 0, 1, 2, ...}
= the set of all whole numbers (positive, negative, 0)

Addition:

$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

Will add video clips when video becomes available.

Formal sum:

4 cone flower + 2 rose

+ 3 cone flower + 1 rose

= 7 cone flower + 3 rose

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where n_i are integers.

Example: $\mathbf{Z}[x_1, x_2]$

$$4x_1 + 2x_2$$

$$x_1 - 2x_2$$

$$-3x_1$$

$$kx_1 + nx_2$$

$\mathbf{Z}$ = The set of integers = { ..., -2, -1, 0, 1, 2, ...}

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$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i are integers.

Example: **Z**[, ]

4  + 2 

 – 2 

-3 

k  + n 

Z = The set of integers = { …, -2, -1, 0, 1, 2, …}

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$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

Example: $\mathbf{Z}[x_1, x_2]$

$$(4x_1 + 2x_2) + (3x_1 + x_2) = 7x_1 + 3x_2$$

$$(4x_1 + 2x_2) + (x_1 - 2x_2) = 5x_1$$

Addition:

$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

Example: $\mathbf{Z}$[, x_2]

$(4$$+ 2x_2) + (3$$+ x_2) = 7$$+ 3x_2$

$$(4x_1 + 2x_2) + (x_1 - 2x_2) = 5x_1$$

Addition:

$$(n_1x_1 + n_2x_2 + \dots + n_kx_k) + (m_1x_1 + m_2x_2 + \dots + m_kx_k)$$

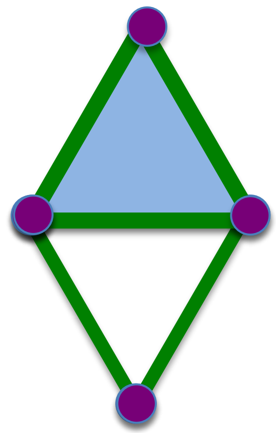
$$= (n_1 + m_1)\,x_1 + (n_2 + m_2)x_2 + \dots + (n_k + m_k)x_k$$

Example: $\mathbf{Z}[x_1, x_2]$

$$(4x_1 + 2x_2) + (3x_1 + x_2) = 7x_1 + 3x_2$$

$$(4x_1 + 2x_2) + (x_1 - 2x_2) = 5x_1$$

Example:



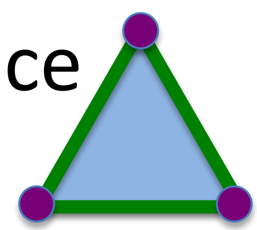
4 vertices + 5 edges + 1 faces

4v + 5e + f.

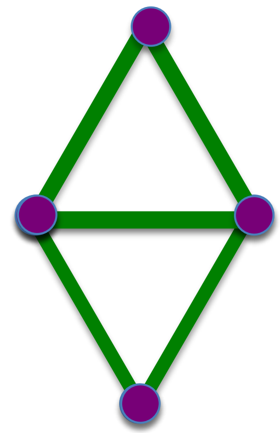
v = vertex

e = edge

f = face



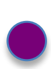
Example 2:



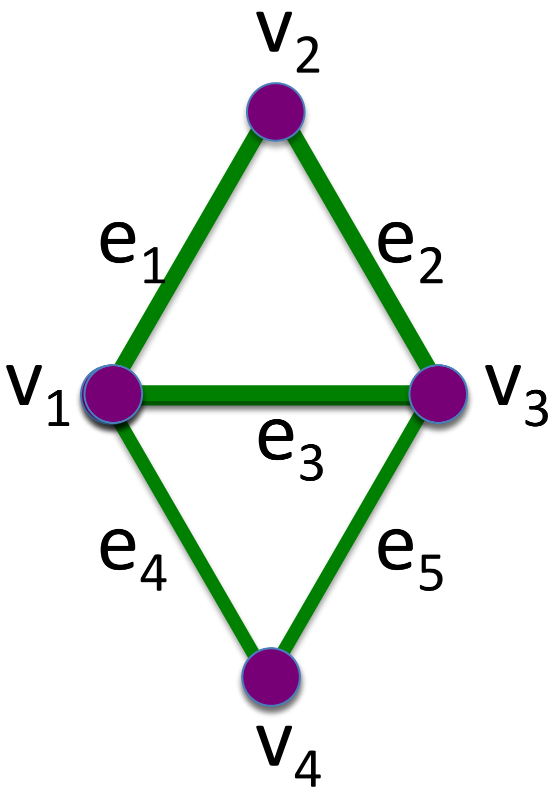
4 vertices + 5 edges

4v + 5e

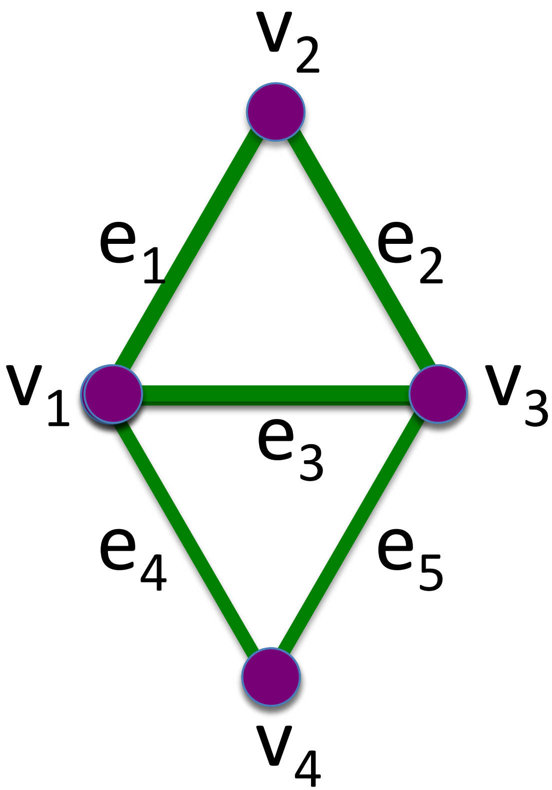
v = vertex



e = edge

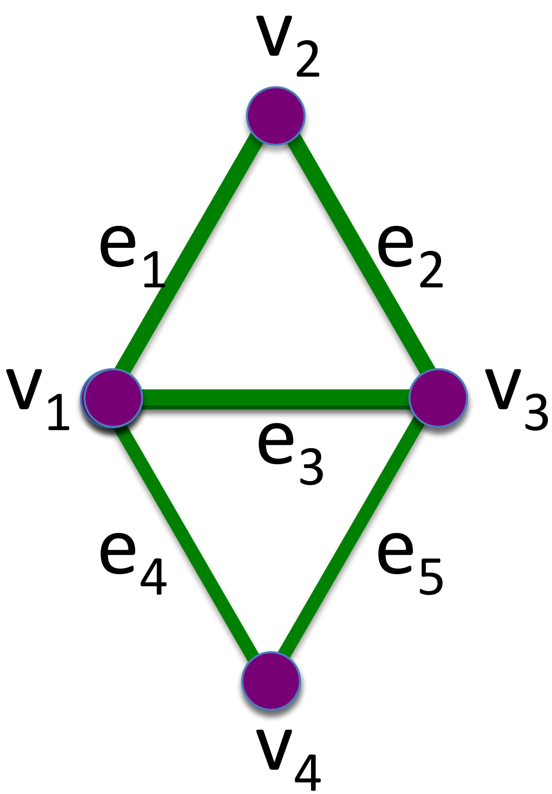


$$v_1 + v_2 + v_3 + v_4 + e_1 + e_2 + e_3 + e_4 + e_5$$



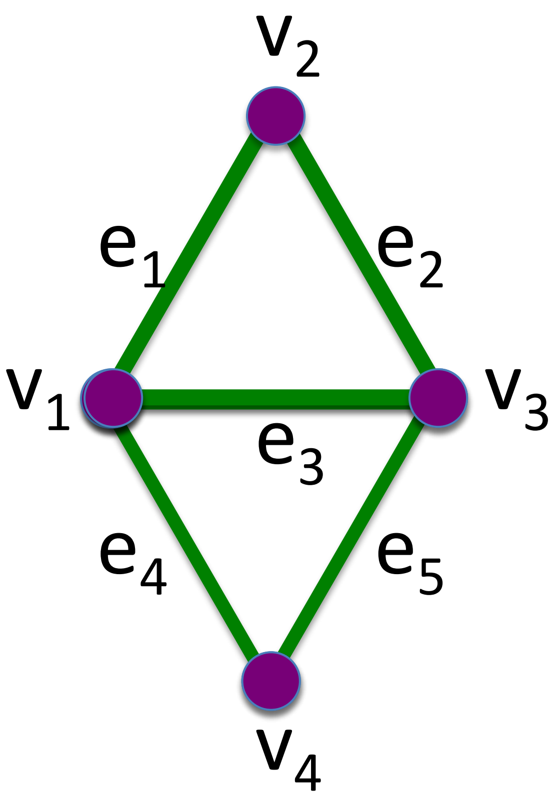
$v_1 + v_2 + v_3 + v_4$ in $\mathbf{Z}[v_1, v_{2,}, v_3, v_4]$

$e_1 + e_2 + e_3 + e_4 + e_5$ in $\mathbf{Z}[e_1, e_{2,}, e_3, e_4, e_5]$



Note that $e_1 + e_2 + e_3$ is a cycle.

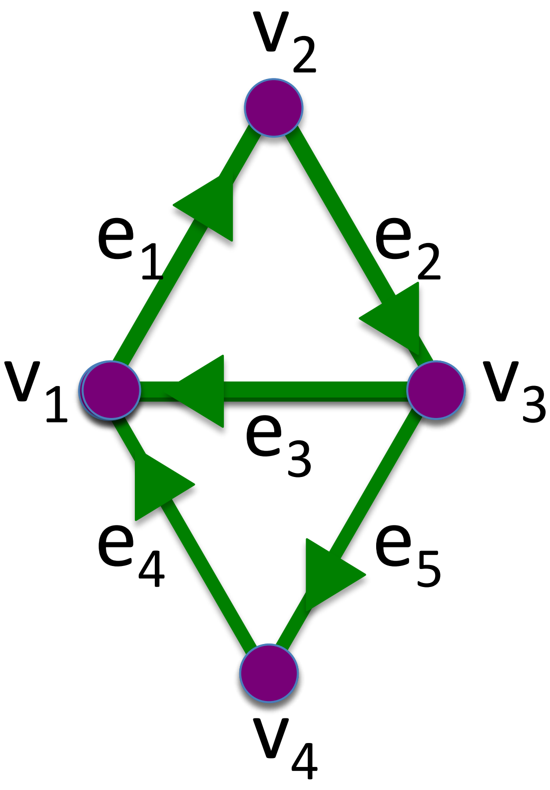
Technical note: In graph theory, the cycle also includes vertices. I.e, this cycle in graph theory is the path v_1, e_1, v_2, e_2, v_3, e_3, $v_{1,}$. Since we are interested in simplicial complexes (see later lecture), we only need the edges, so $e_1 + e_2 + e_3$ is a cycle.



Note that $e_1 + e_2 + e_3$ is a cycle.

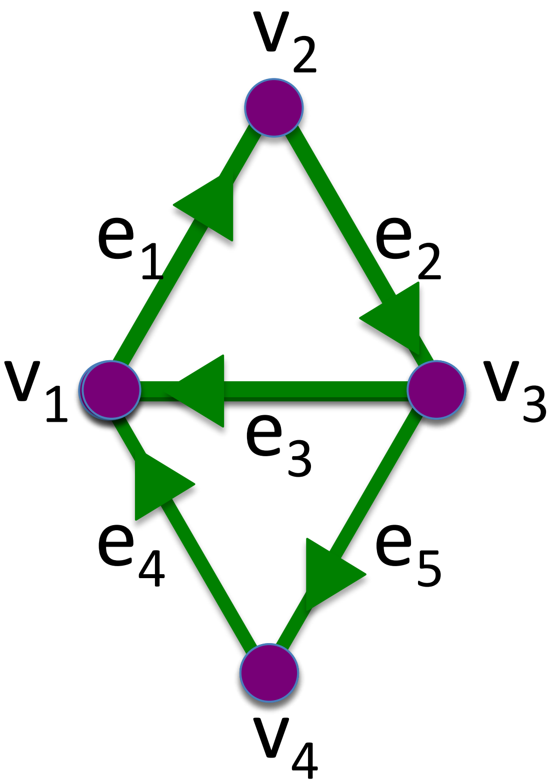
Note that $e_3 + e_4 + e_5$ is a cycle.

Technical note: In graph theory, the cycle also includes vertices. I.e, the cycle in graph theory is the path v_1, e_1, v_2, e_2, v_3, e_3, $v_{1,}$. Since we are interested in simplicial complexes (see later lecture), we only need the edges, so $e_1 + e_2 + e_3$ is a cycle.



Note that $e_1 + e_2 + e_3$ is a cycle.

Note that $-e_3 + e_5 + e_4$ is a cycle.



Note that $e_1 + e_2 + e_3$ is a cycle.

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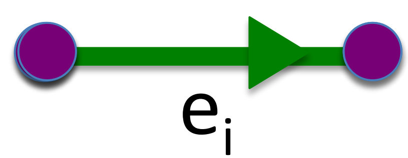
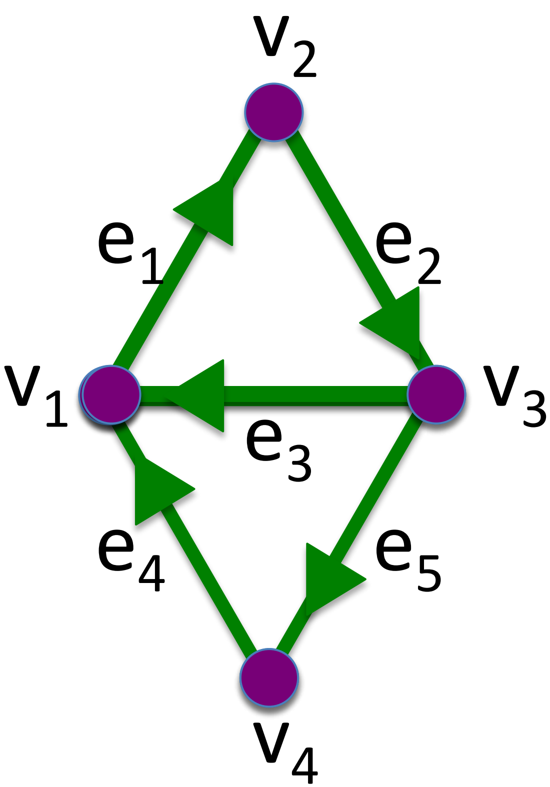
Objects: oriented edges





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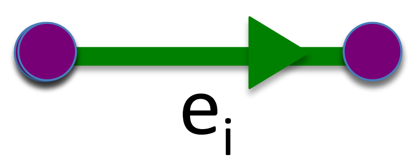
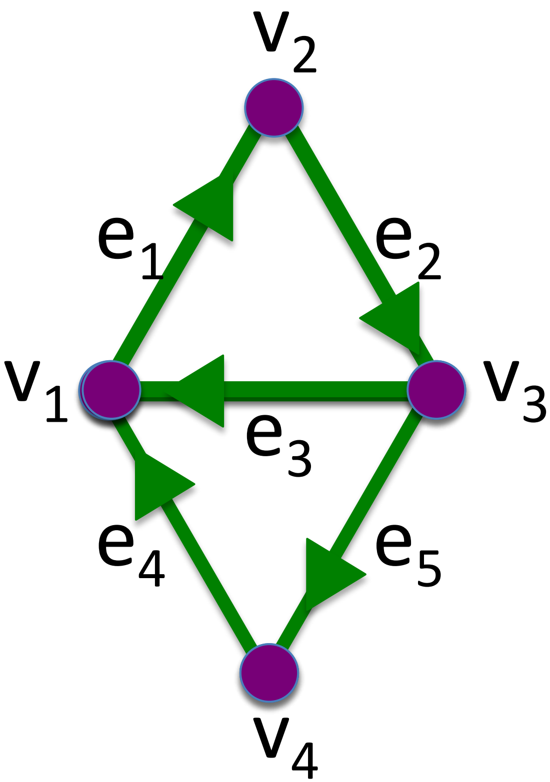
Objects: oriented edges



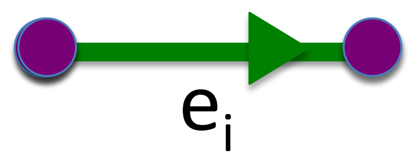
in $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$

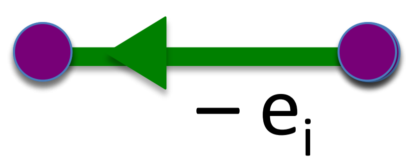


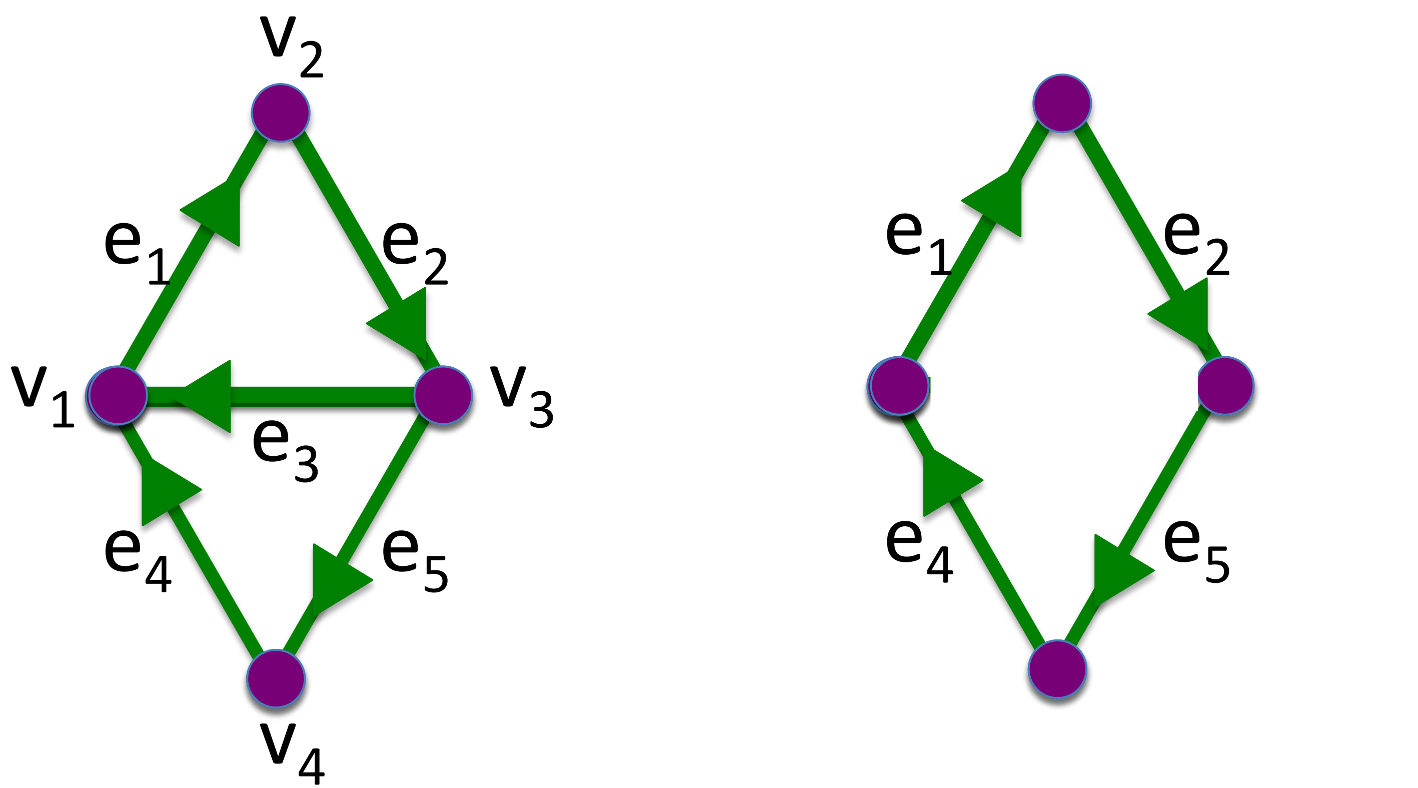
Note that $e_1 + e_2 + e_3$ is a cycle.

Note that $-e_3 + e_5 + e_4$ is a cycle.

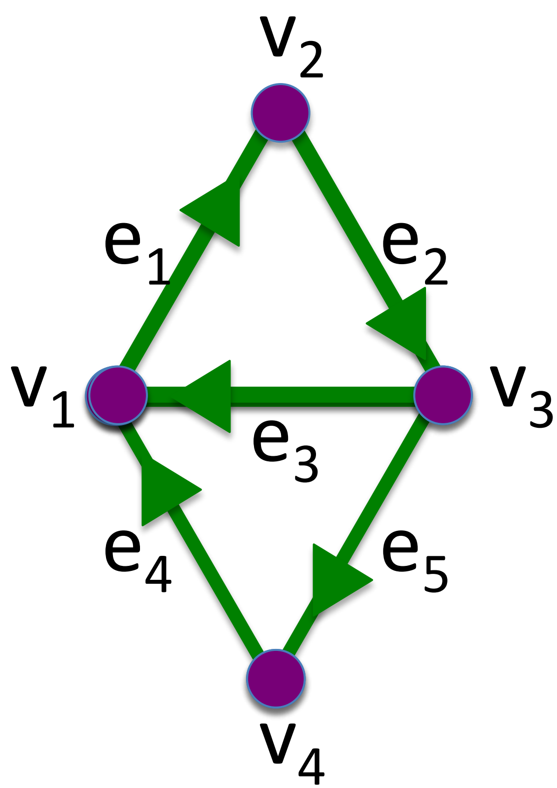
Objects: oriented edges in $\mathbf{Z}[e_1, e_2, e_3, e_4, e_5]$

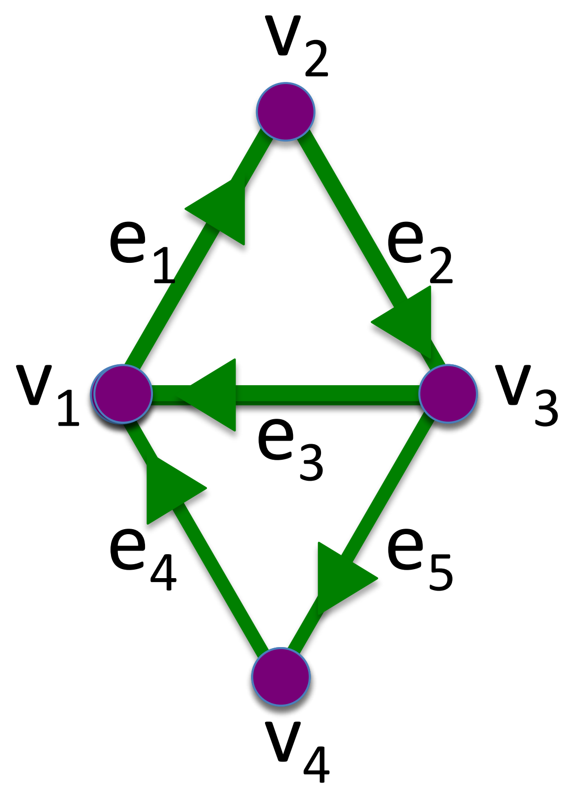






$$(e_1 + e_2 + e_3) + (-e_3 + e_5 + e_4) = e_1 + e_2 + e_5 + e_4$$





$$e_1 + e_2 + e_3 + e_1 + e_2 + e_5 + e_4 = 2e_1 + 2e_2 + e_3 + e_4 + e_5$$

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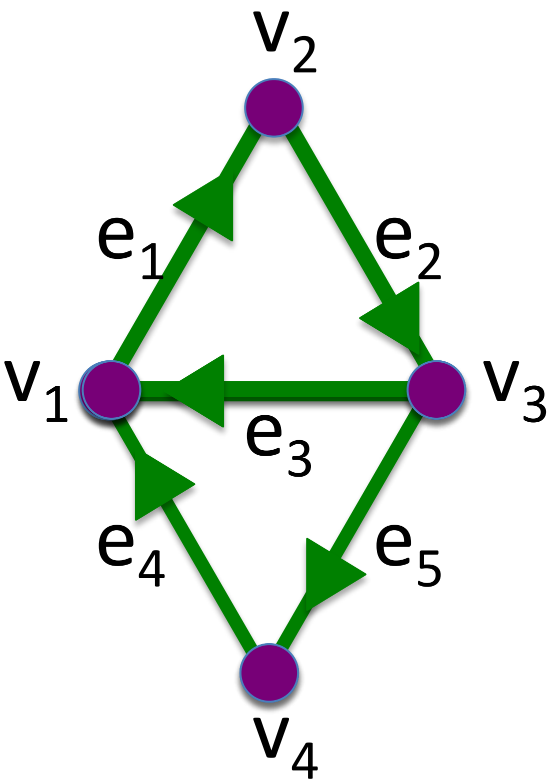
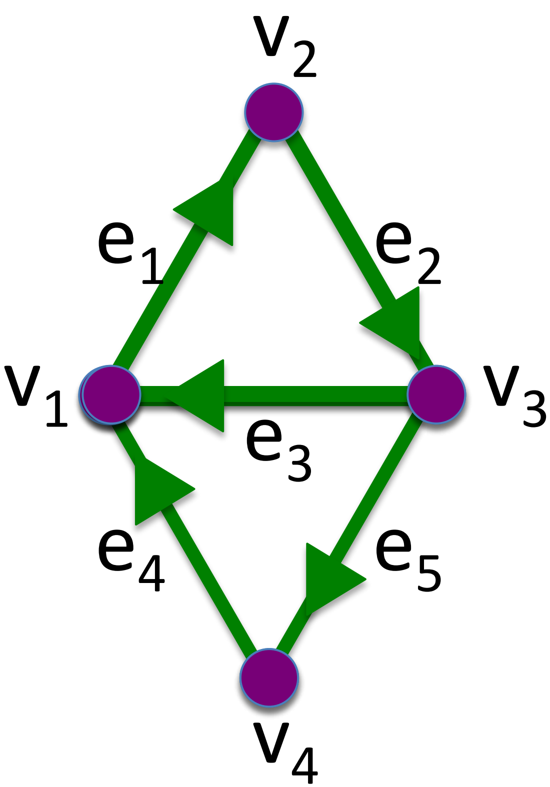


The boundary of $e_1 = v_2 - v_1$



The boundary of $e_1 = v_2 - v_1$

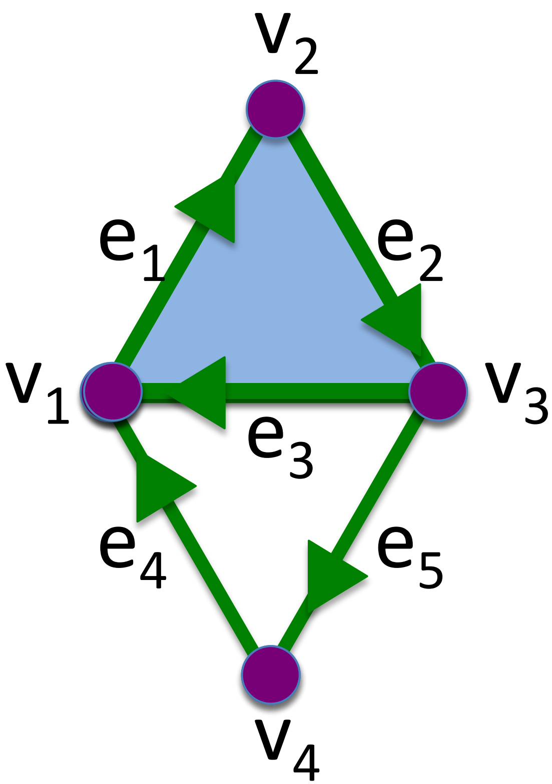
The boundary of $e_2 = v_3 - v_2$

The boundary of $e_3 = v_1 - v_3$

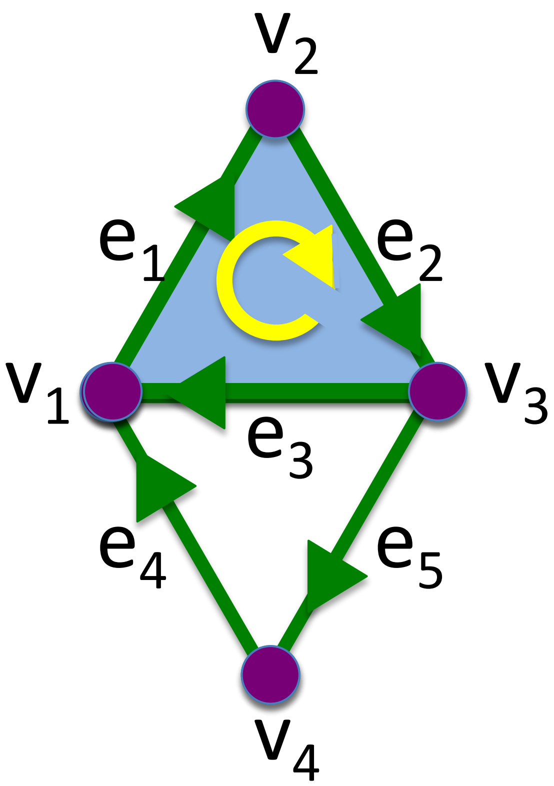
The boundary of $e_1 + e_2 + e_3$

$$= v_2 - v_1 + v_3 - v_2 + v_1 - v_3 = 0$$

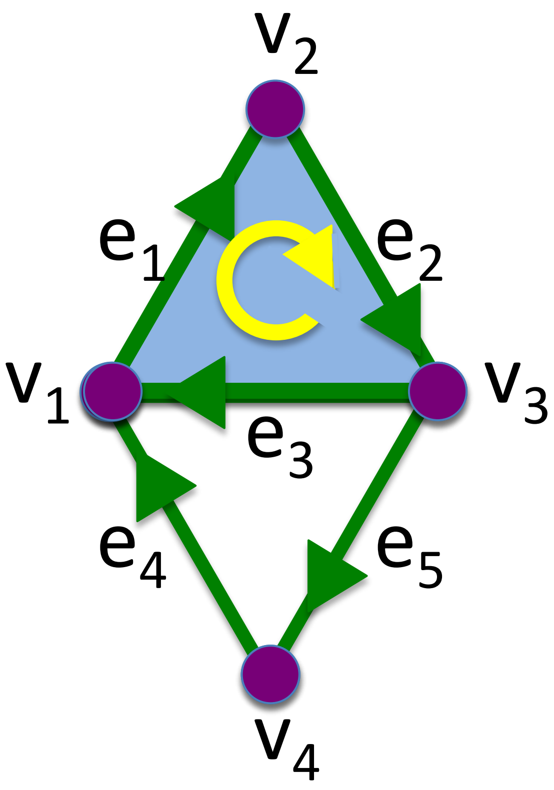
Add a face

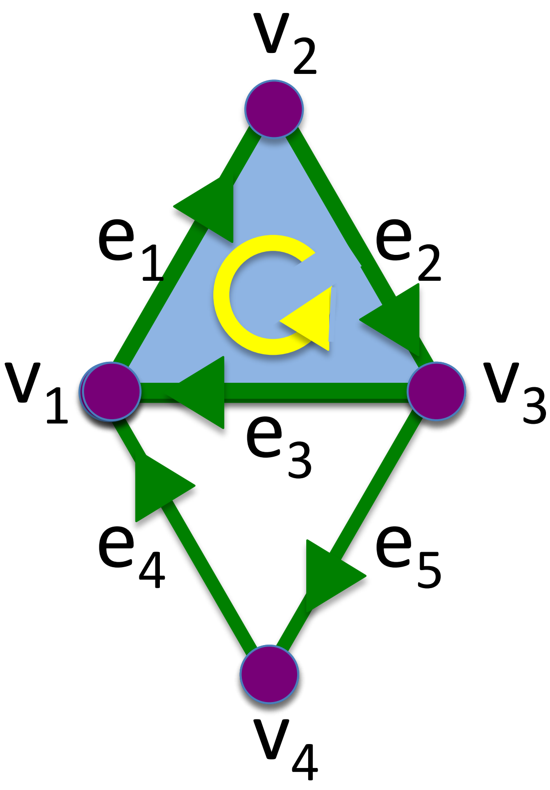


Add an oriented face



Add an oriented face





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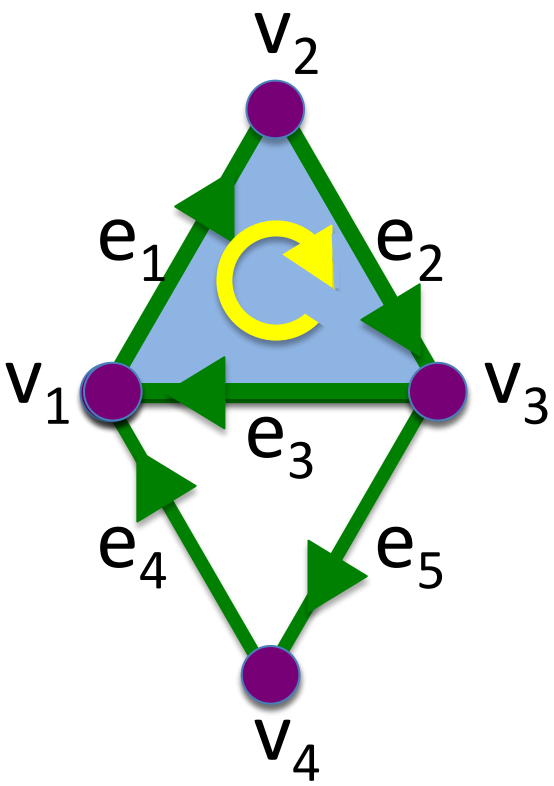
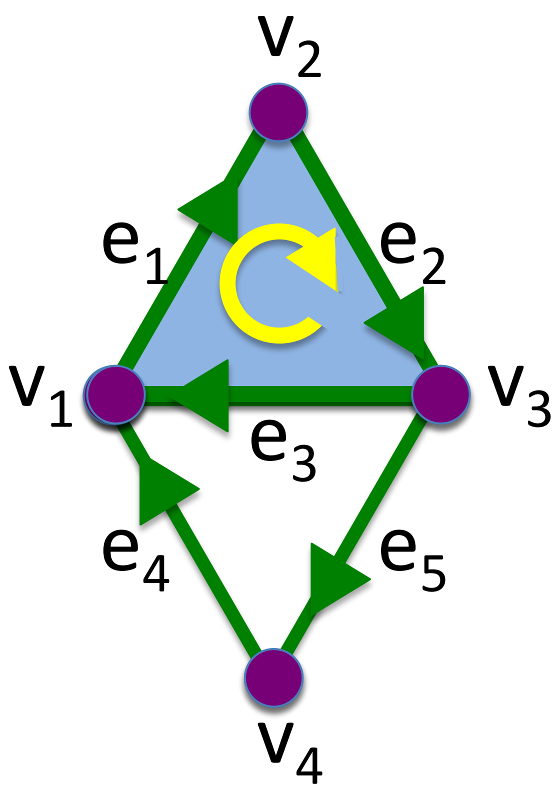
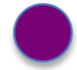


Note that the boundary of this face is the cycle

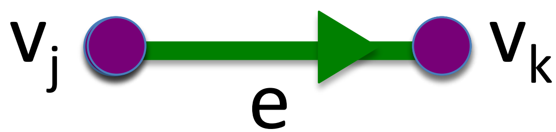
$$e_1 + e_2 + e_3$$

Simplicial complex



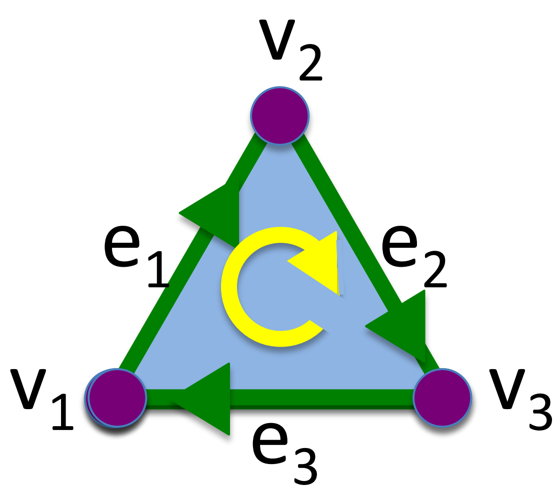
0-simplex = vertex = v 

1-simplex = oriented edge = (v_j, v_k)



Note that the boundary of this edge is $v_k - v_j$

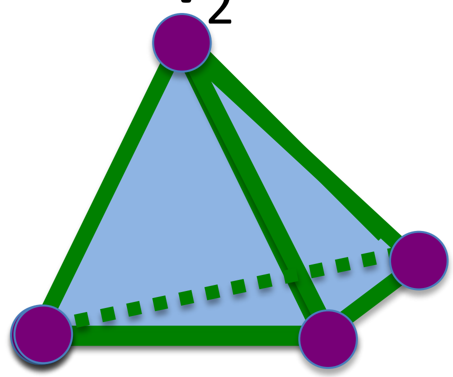
2-simplex = oriented face = (v_i, v_j, v_k)



Note that the boundary of this face is the cycle

$$e_1 + e_2 + e_3$$

3-simplex = (v_1, v_2, v_3, v_4) = tetrahedron



4-simplex = (v_1, v_2, v_3, v_4, v_5)