Lecture 1: The Euler characteristic

of a series of preparatory lectures for the Fall 2013 online course MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Target Audience: Anyone interested in **topological data analysis** including graduate students, faculty, industrial researchers in bioinformatics, biology, computer science, cosmology, engineering, imaging, mathematics, neurology, physics, statistics, etc.

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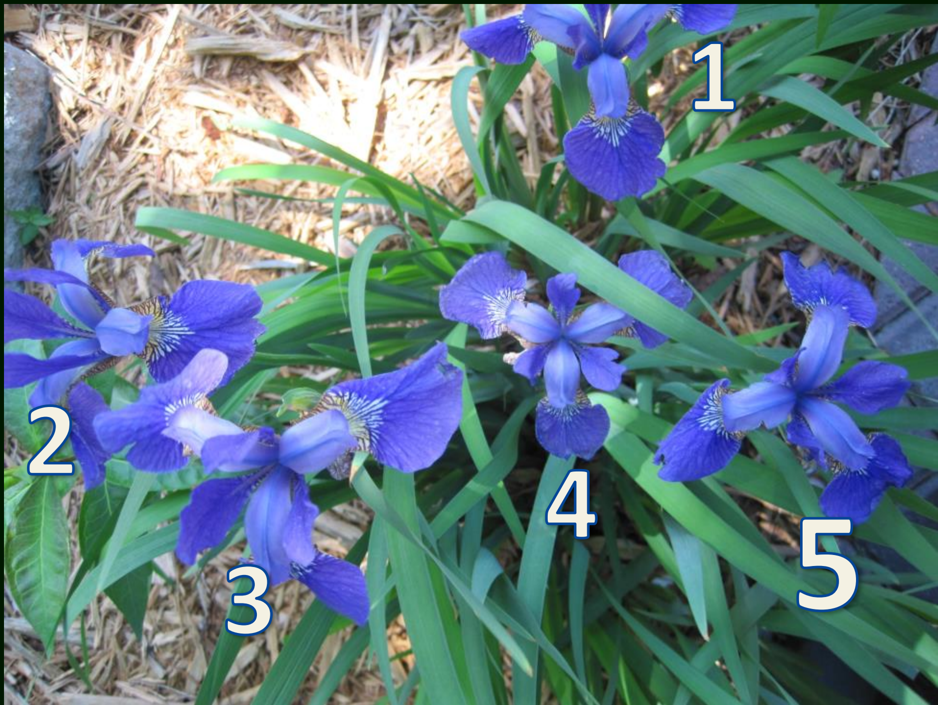
Mathematics Department/Applied Mathematical & Computational Sciences
University of Iowa

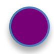
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

Counting



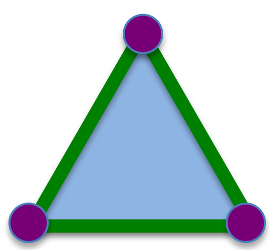
We wish to count:

vertex

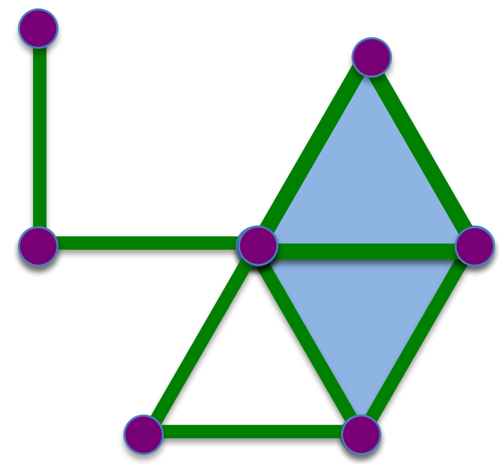


edge

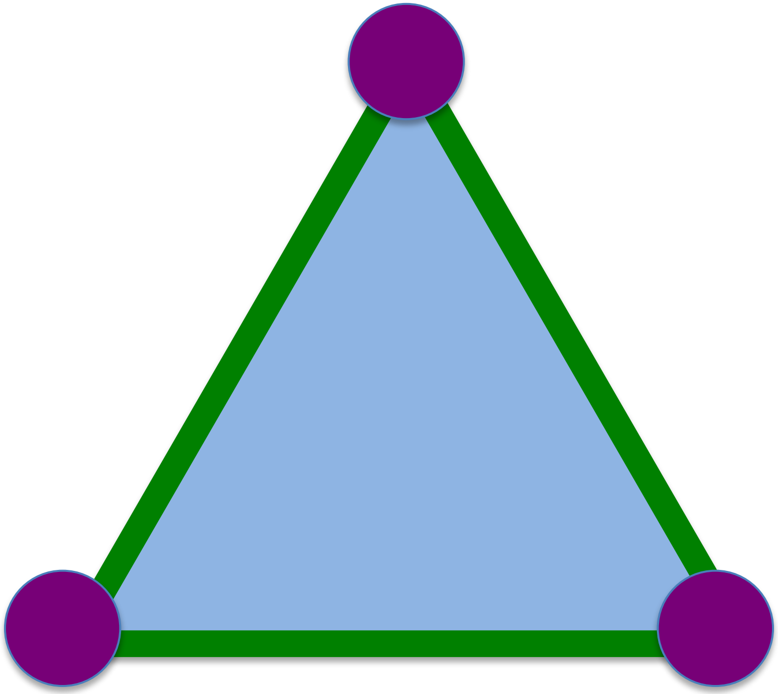
face



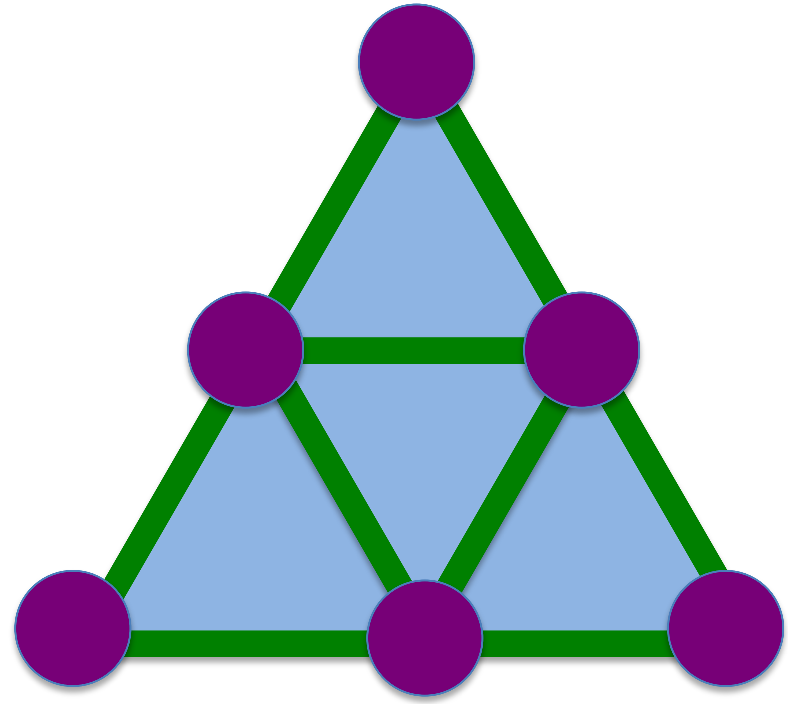
Example:



7 vertices,
9 edges,
2 faces.



3 vertices,
3 edges,
1 face.



6 vertices,
9 edges,
4 faces.

The formula:

Euler characteristic

Euler characteristic (simple form):

χ = number of vertices – number of edges + number of faces

Or in short-hand,

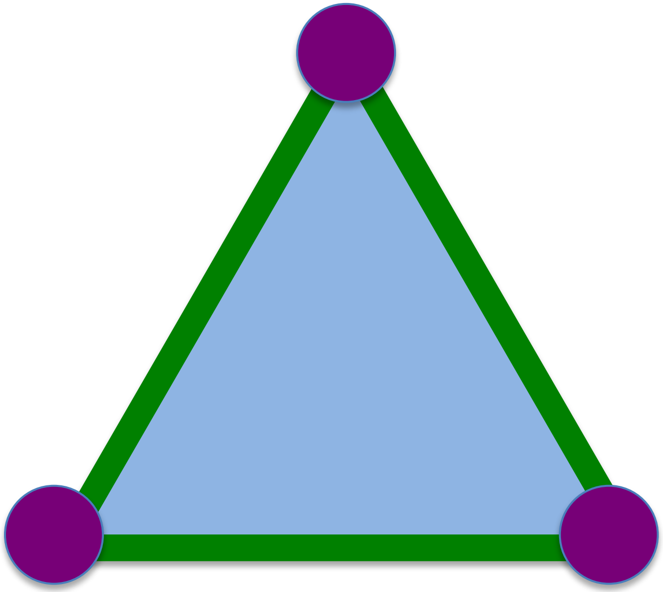
$$\chi = |V| - |E| + |F|$$

where V = set of vertices

E = set of edges

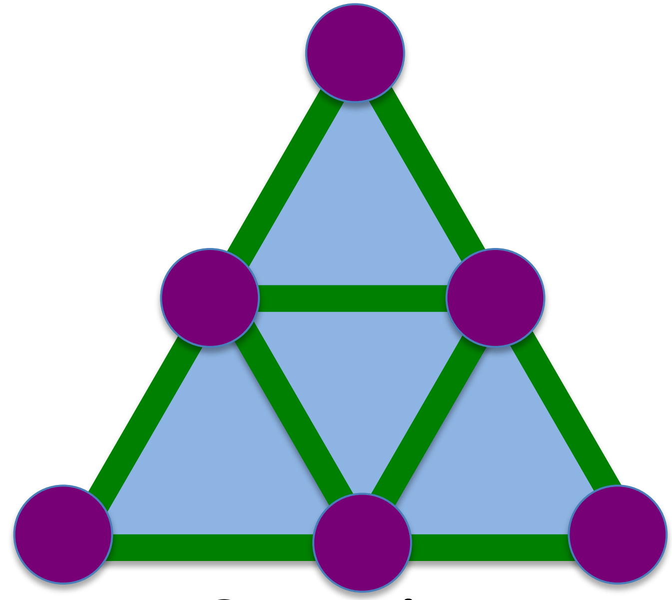
F = set of faces

& the notation |X| = the number of elements in the set X.



3 vertices,
3 edges,
1 face.

$\chi = |V| - |E| + |F| =$
$3 - 3 + 1 = 1$



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9 edges,
4 faces.

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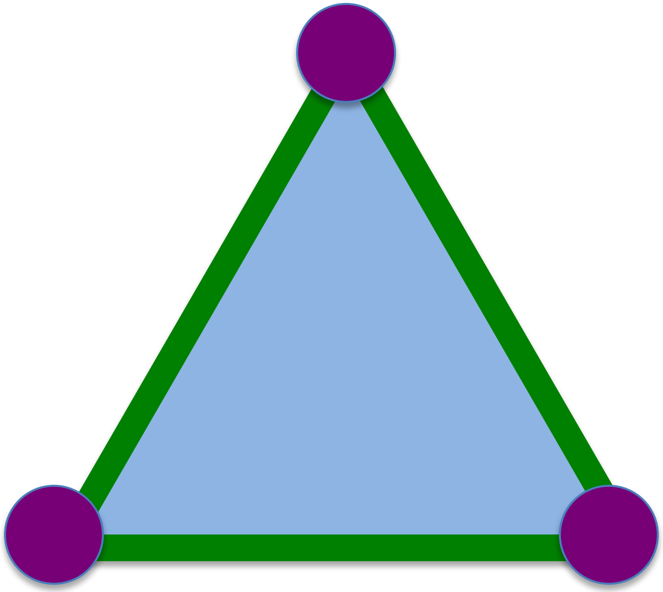
$6 - 9 + 4 = 1$

Note: 3 – 3 + 1 = **1** = 6 – 9 + 4



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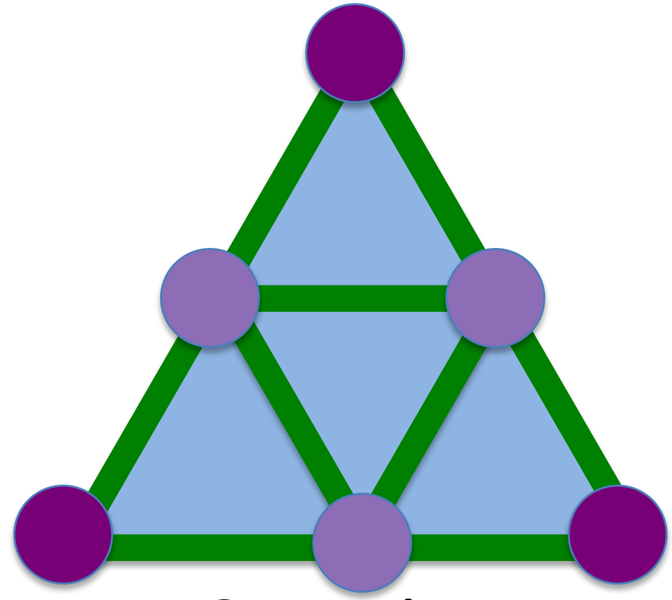

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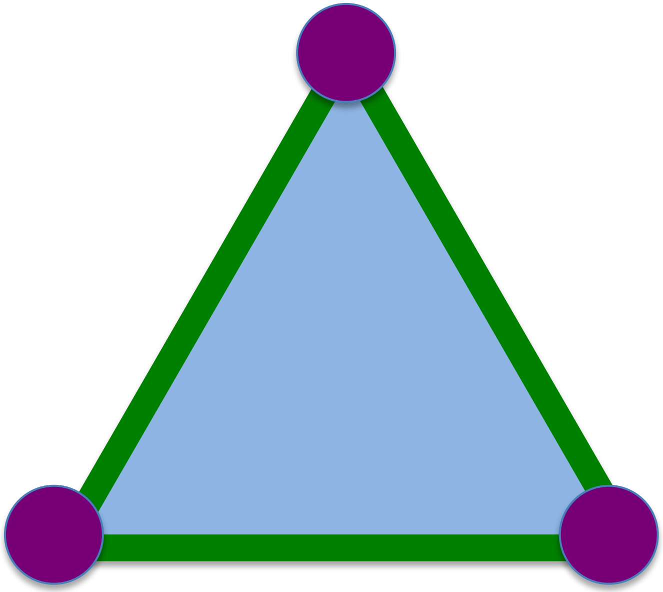

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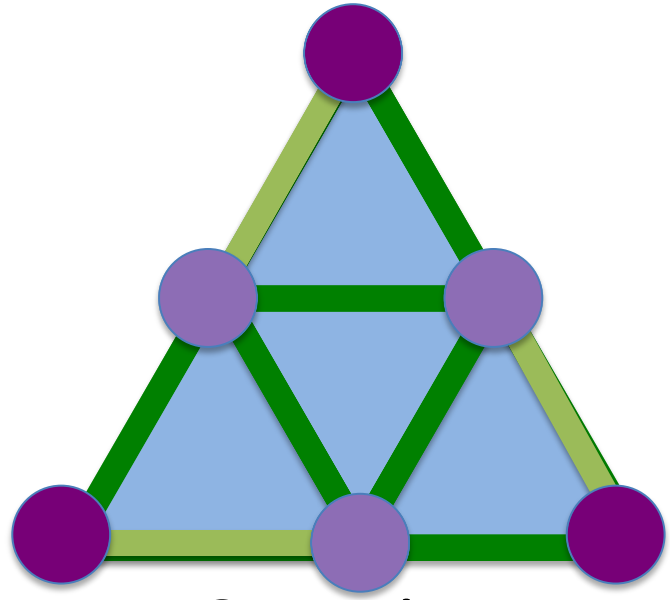

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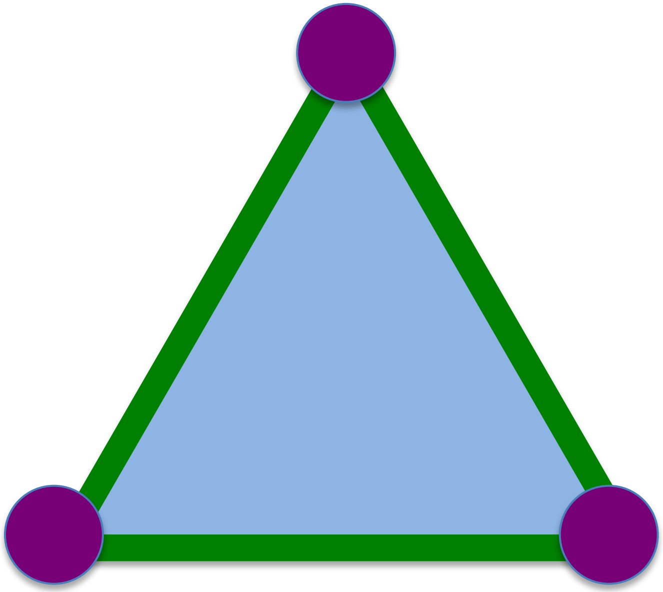

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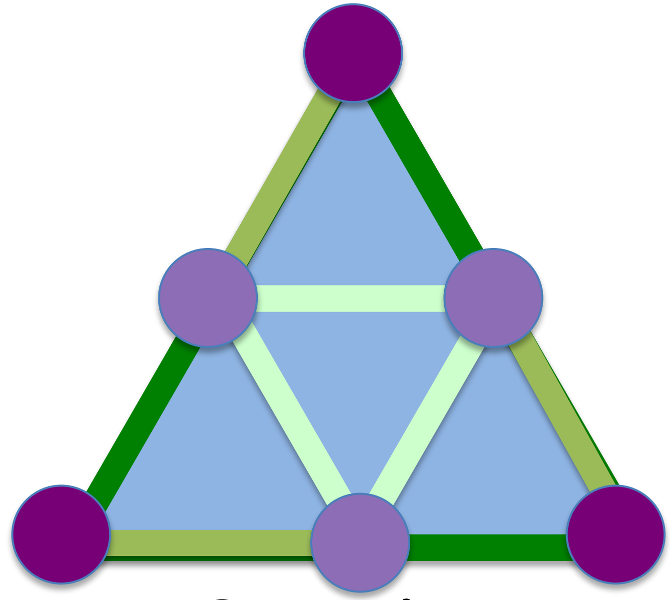

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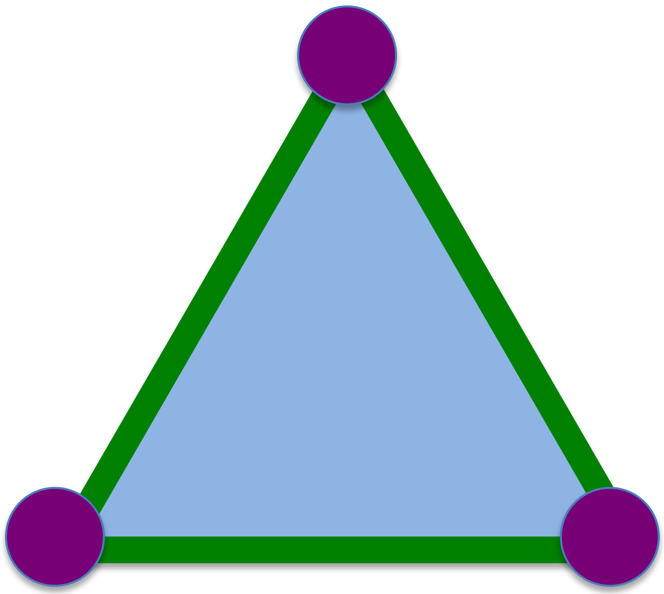

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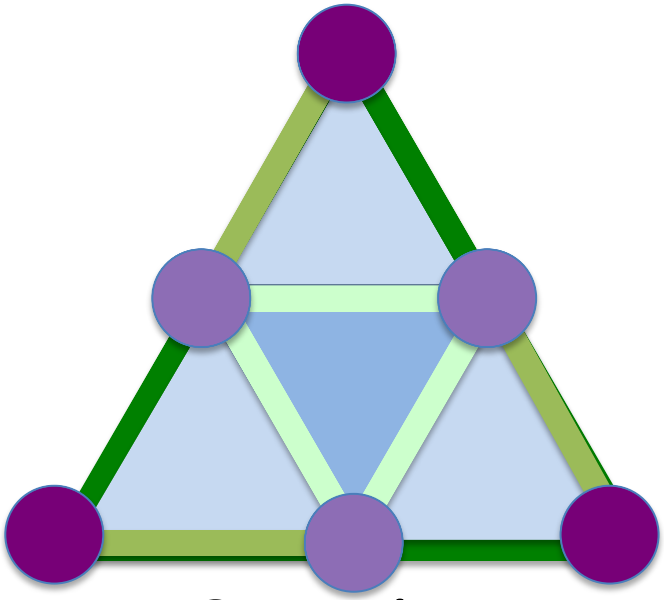

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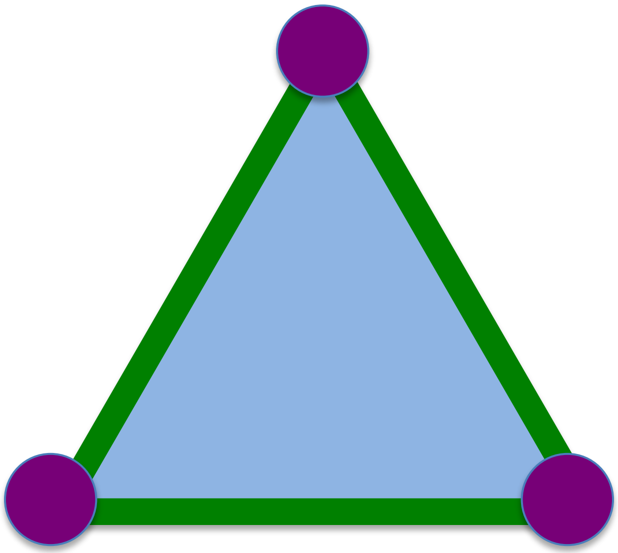
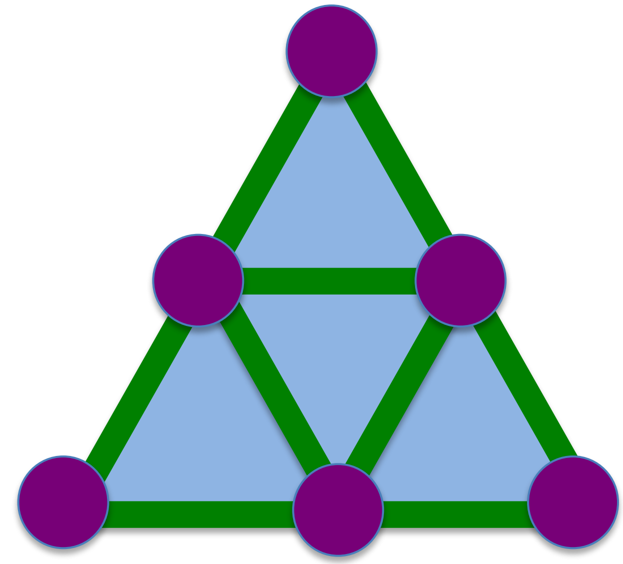
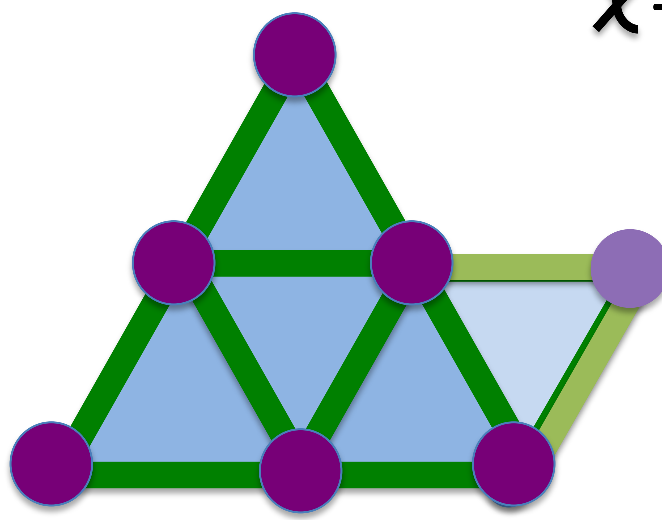


$\chi = 3 - 3 + 1 = 1$

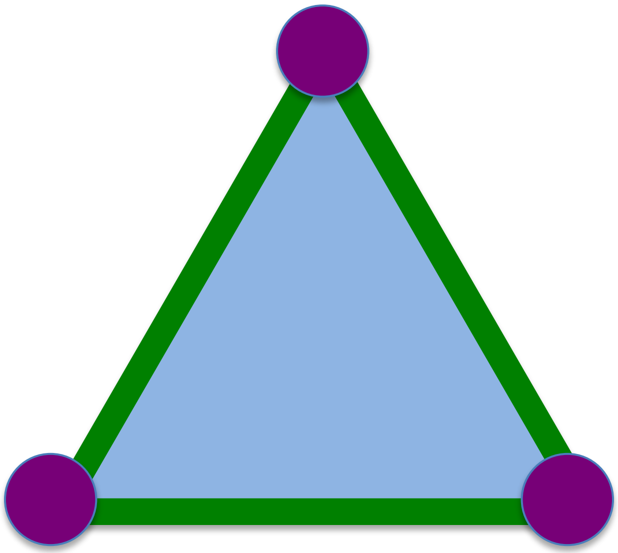


$\chi = 6 - 9 + 4 = 1$

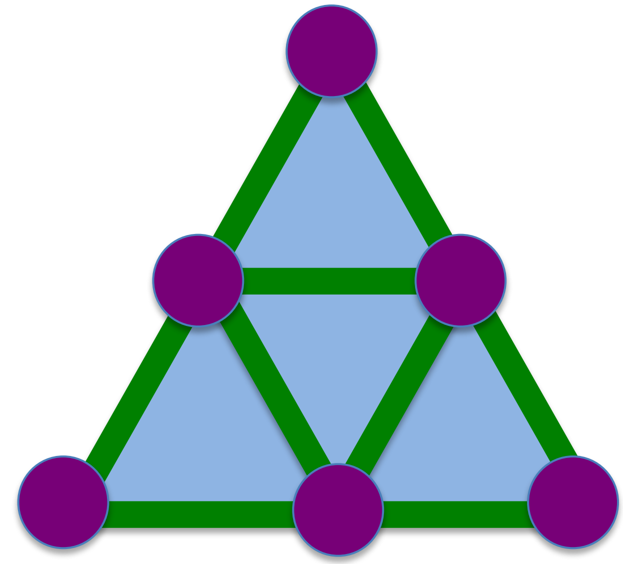


$\chi = 7 - 11 + 5 = 1$

$\chi = |V| - |E| + |F|$



$\chi = 3 - 3 + 1 = 1$



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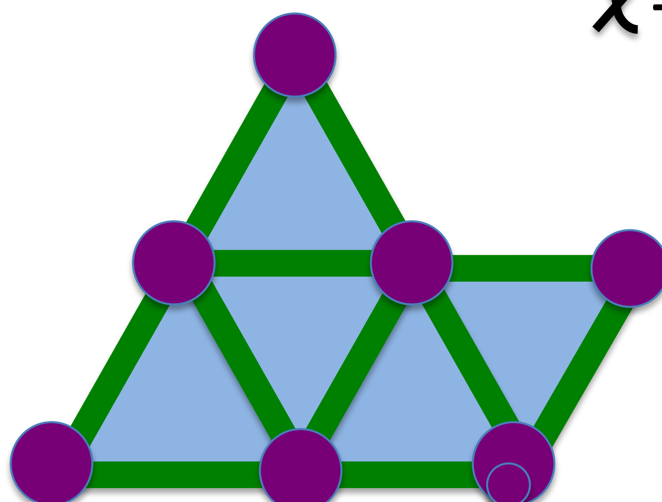
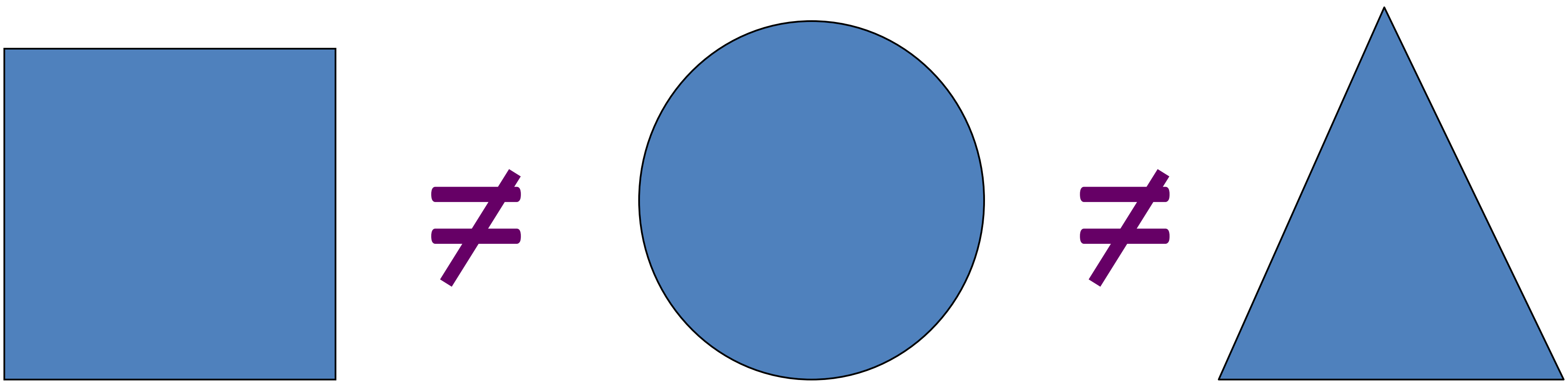


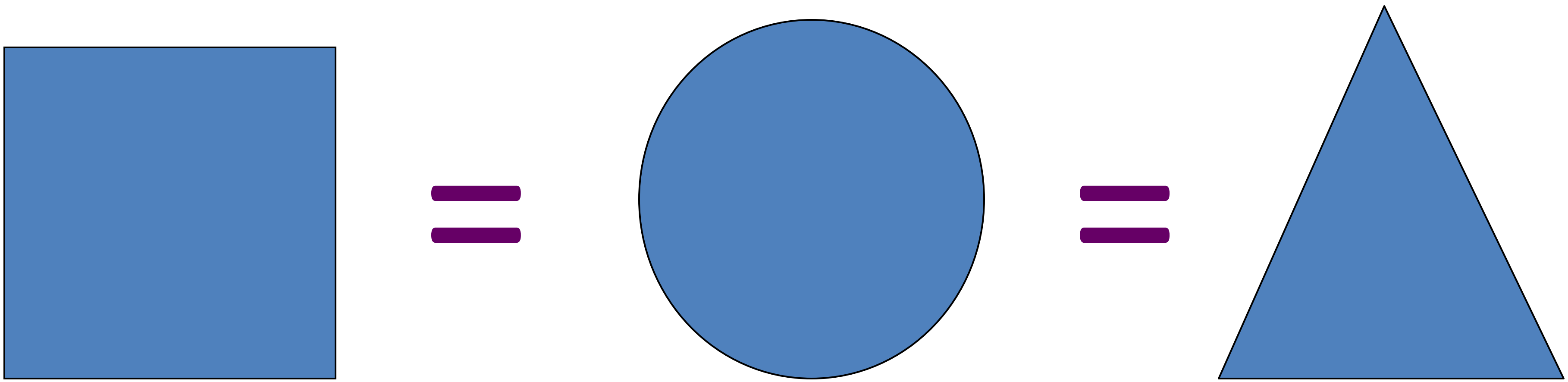
$\chi = 7 - 11 + 5 = 1$

$\chi = |V| - |E| + |F|$

Geometry

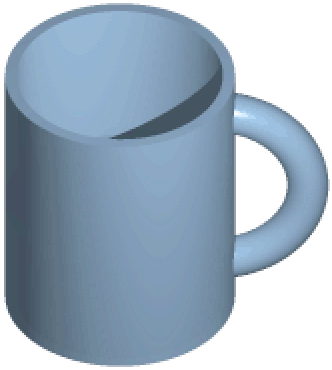


Topology



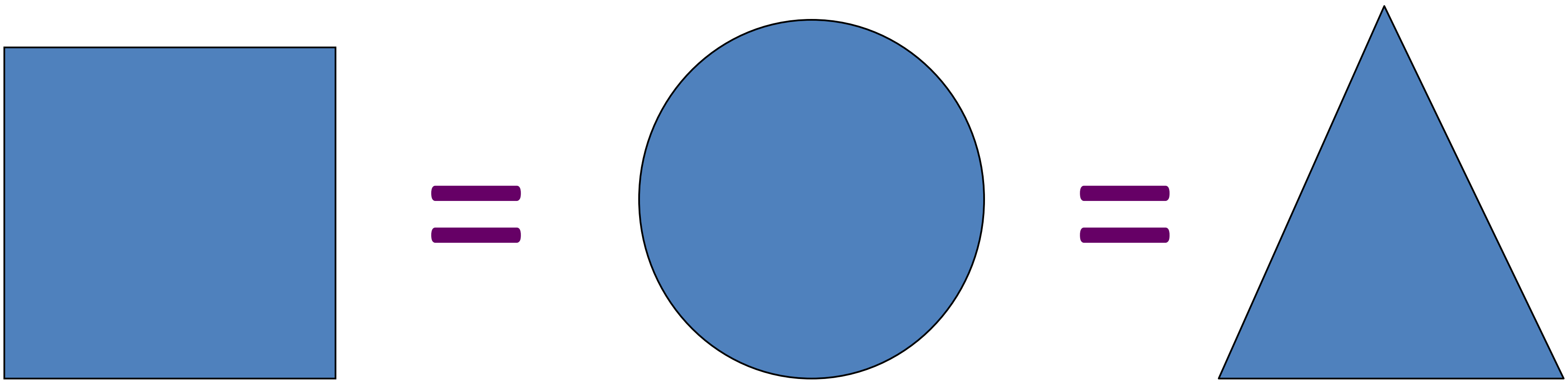
Video Insert illustrating topology

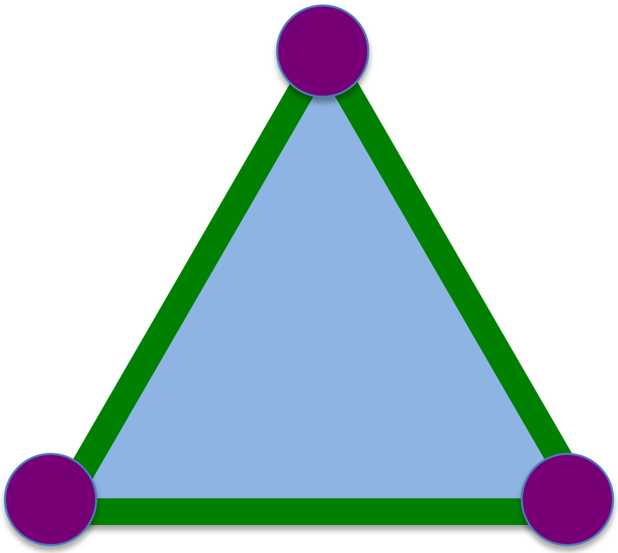
Note a coffee cup is topologically equivalent to a donut



gif from https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif

Topology





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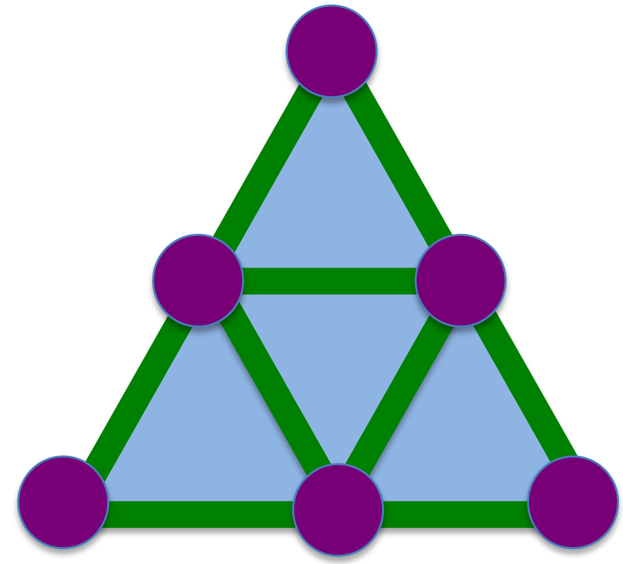

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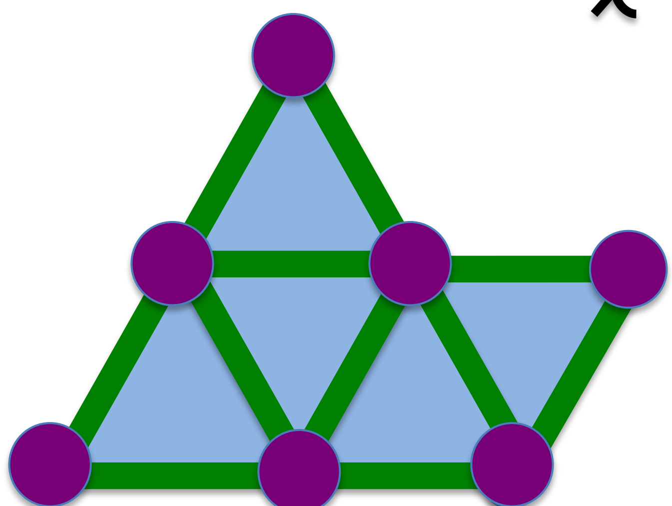
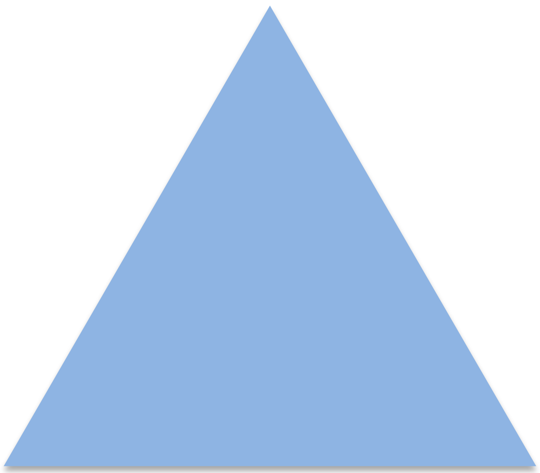


$\chi = 7 - 11 + 5 = 1$

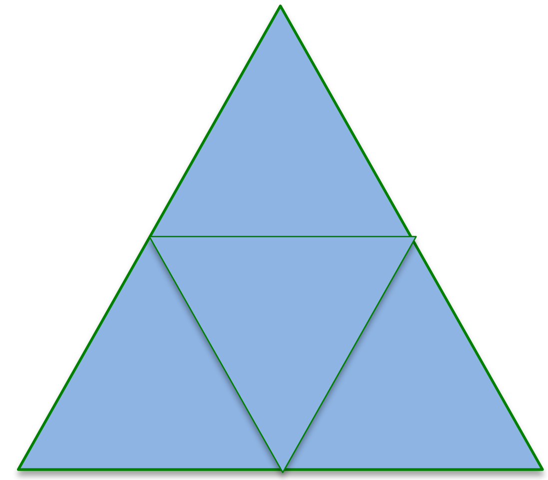
$\chi = |V| - |E| + |F|$

The Euler characteristic is a topological invariant

That means that if two objects are topologically the same, they have the same Euler characteristic.



$\chi = 1$



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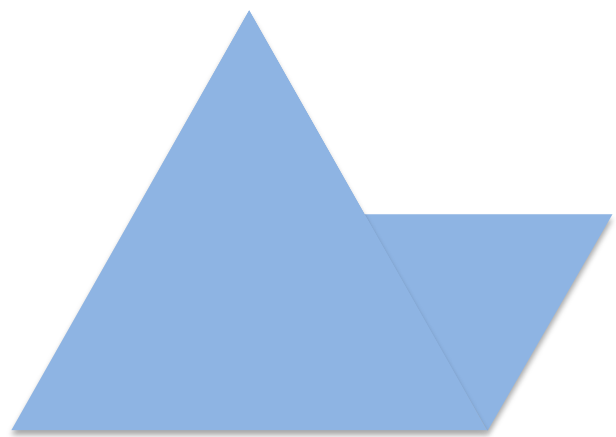

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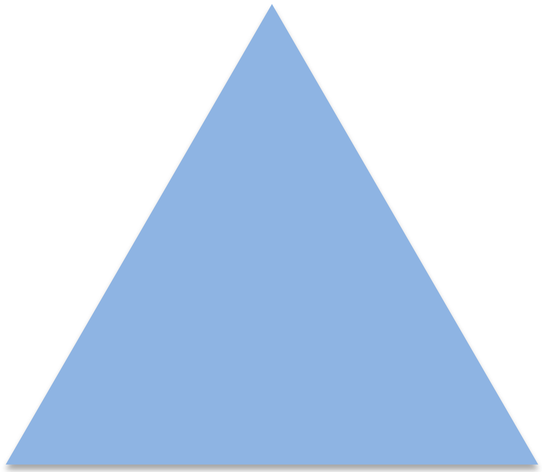

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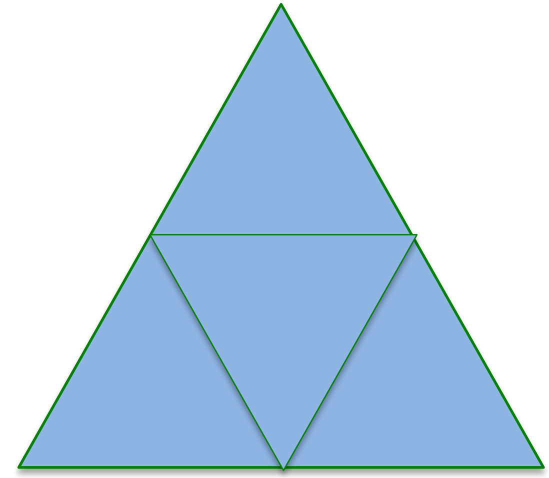

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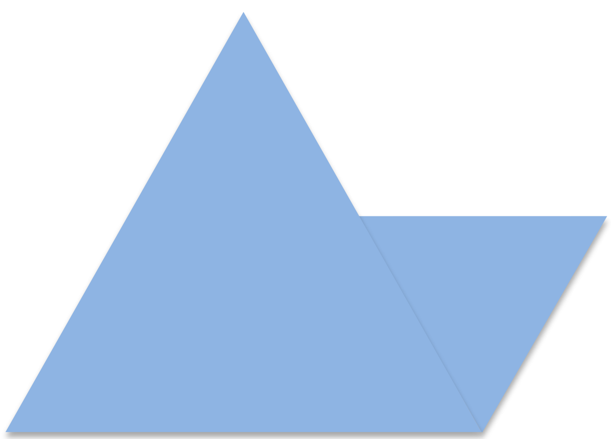

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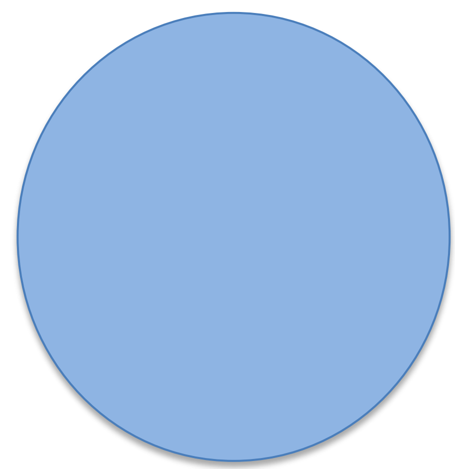

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The Euler characteristic is a topological invariant

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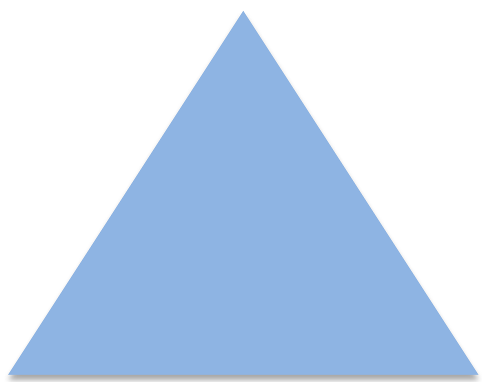
Example:



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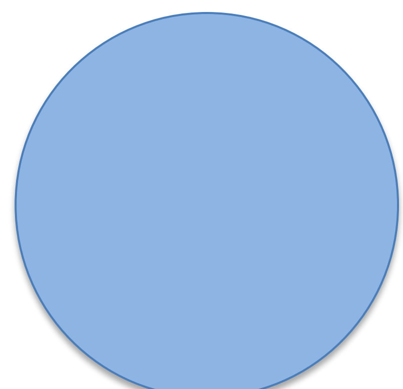
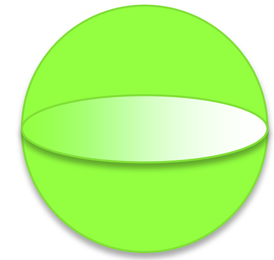
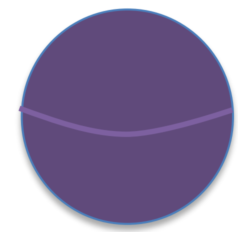
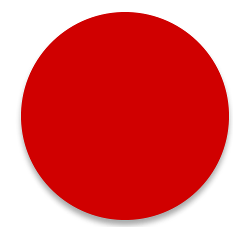


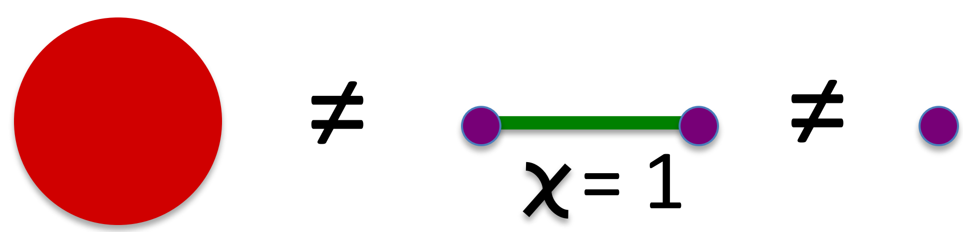
$\chi = 1$

Euler characteristic		
2	sphere $= \{ x \text{ in } R^3 : \|\|x\|\| = 1 \}$	
1	ball $= \{ x \text{ in } R^3 : \|\|x\|\| \leq 1 \}$	
	disk $= \{ x \text{ in } R^2 : \|\|x\|\| \leq 1 \}$	
	closed interval $= \{ x \text{ in } R : \|\|x\|\| \leq 1 \}$	

The Euler characteristic is a topological invariant

That means that if two objects are topologically the same, they have the same Euler characteristic.

But objects with the same Euler characteristic need not be topologically equivalent.



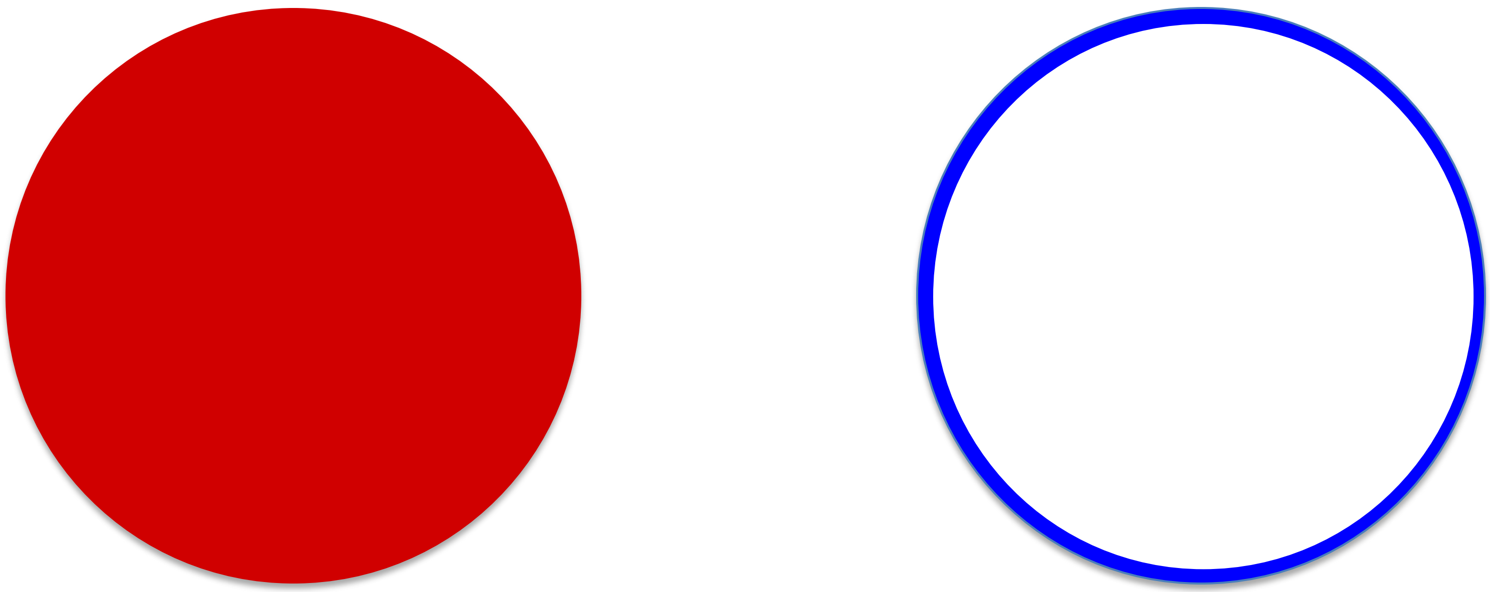
Let R be a subset of X

A *deformation retract* of X onto R is a continuous map $F: X \times [0, 1] \rightarrow X$, $F(x, t) = f_t(x)$ such that

f_0 is the identity map,

$f_1(X) = R$, and

$f_t(r) = r$ for all r in R.



If R is a deformation retract of X, then $\chi(R) = \chi(X)$.

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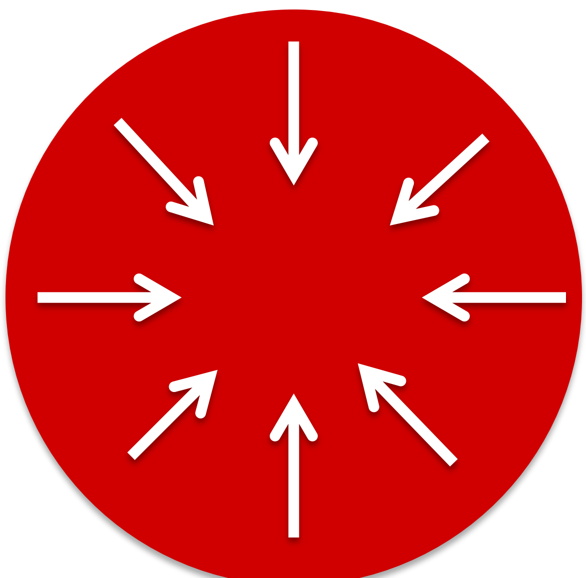
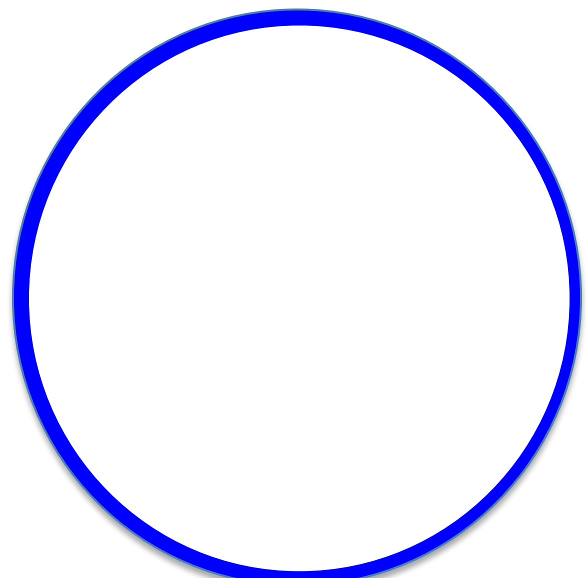
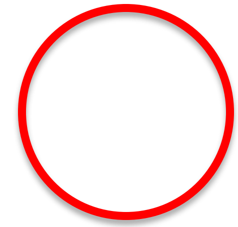


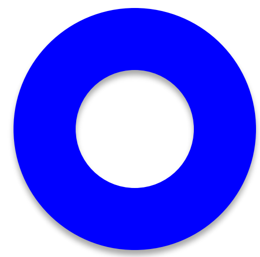


If R is a deformation retract of X, then $\chi(R) = \chi(X)$.

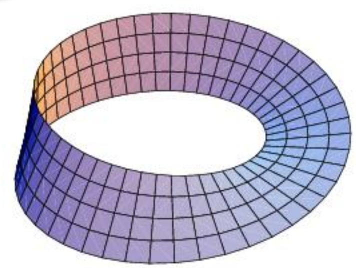
Euler characteristic	
0	S^1 = circle = $\{ x \in R^2 : \|\|x\|\| = 1 \}$



Annulus

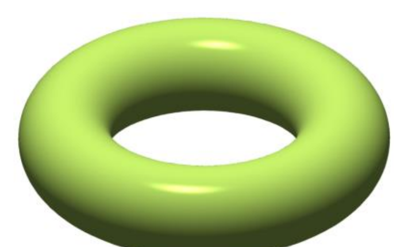


Mobius band

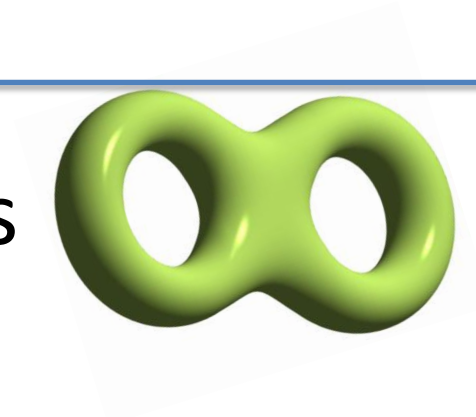
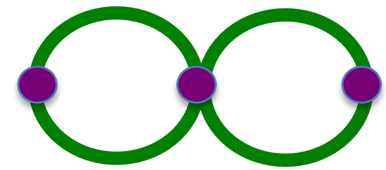
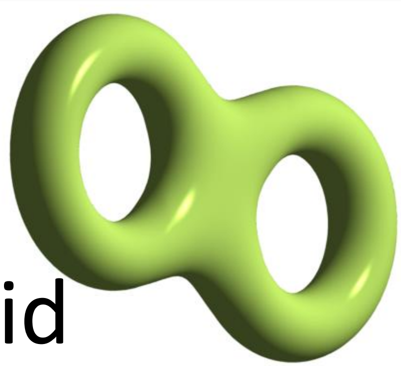


Solid torus = S^1 x disk

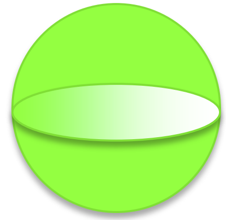
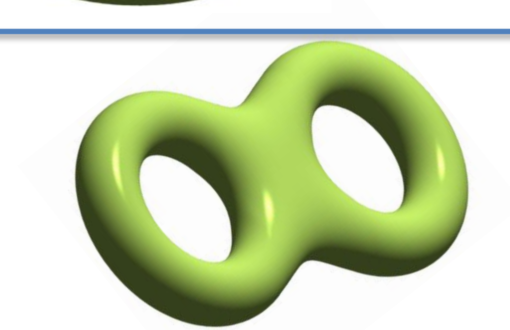
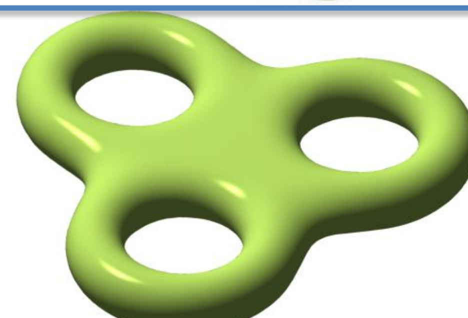
Torus = S^1 x S^1



Mobius band and torus images from https://en.wikipedia.org/wiki/Euler_characteristic

Euler characteristic	
-1	Solid double torus  The graph: 
-2	Double torus = genus 2 torus = boundary of solid double torus 

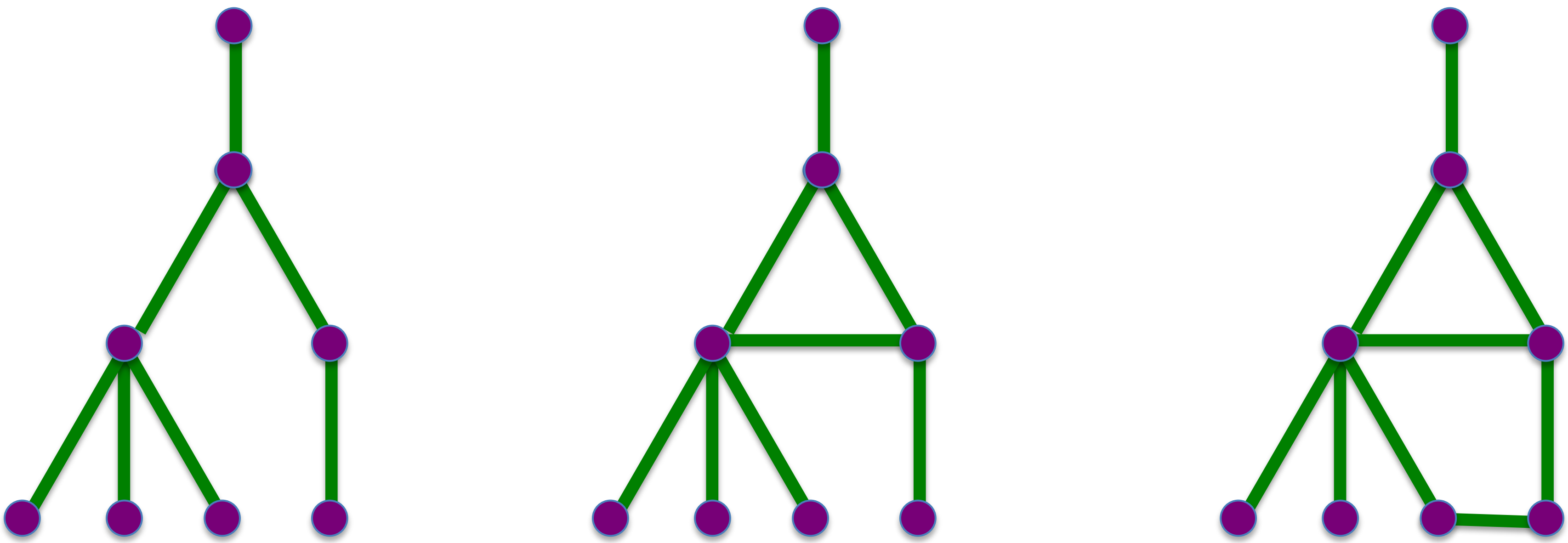
Genus n tori images from https://en.wikipedia.org/wiki/Euler_characteristic

Euler characteristic	2-dimensional orientable surface without boundary
2	sphere 
0	$S^1 \times S^1$ = torus 
-2	genus 2 torus 
-4	genus 3 torus 

Genus n tori images from https://en.wikipedia.org/wiki/Euler_characteristic

Graphs: Identifying Trees

Defn: A *tree* is a connected graph that does not contain a cycle



$\chi = 8 - 7 = 1$ $\chi = 8 - 8 = 0$ $\chi = 8 - 9 = -1$

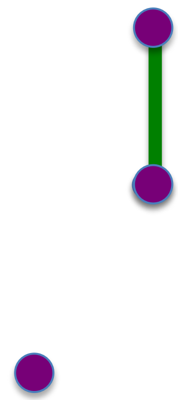
$\chi = |V| - |E| + |F|$

$\chi = 2$

$\chi = |V| - |E| + |F|$

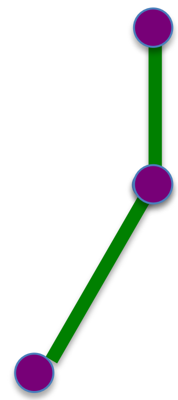
$\chi = 2 - 1 = 1$

$\chi = |V| - |E| + |F|$



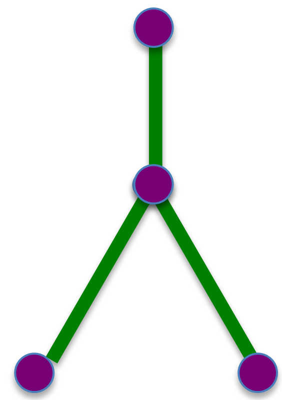
$\chi = 3 - 1 = 2$

$\chi = |V| - |E| + |F|$



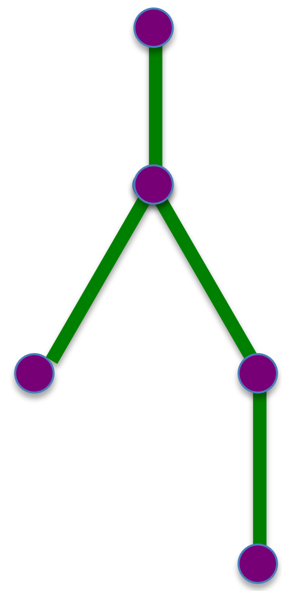
$\chi = 3 - 2 = 1$

$\chi = |V| - |E| + |F|$



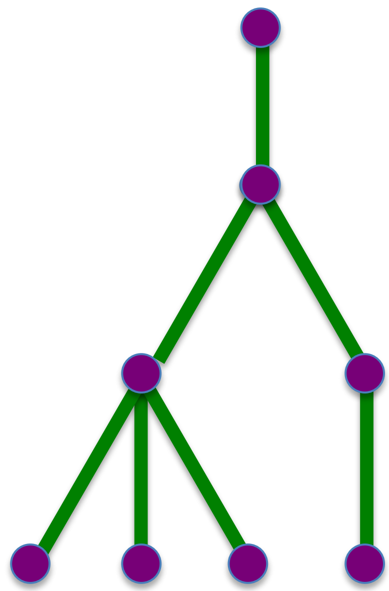
$\chi = 4 - 3 = 1$

$\chi = |V| - |E| + |F|$



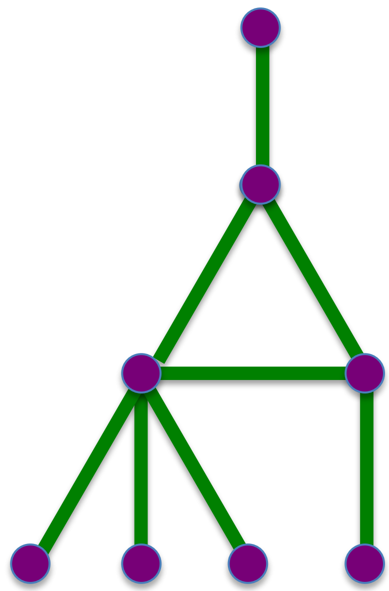
$\chi = 5 - 4 = 1$

$\chi = |V| - |E| + |F|$



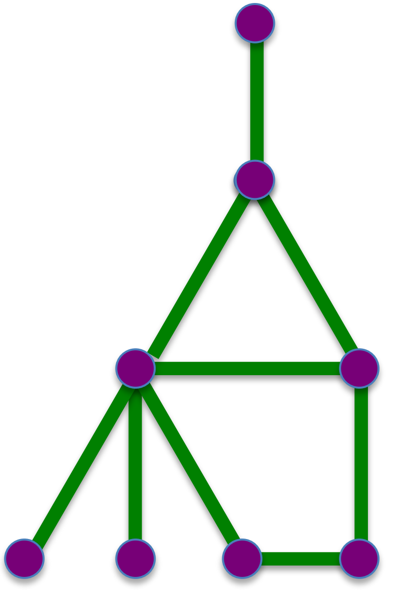
$\chi = 8 - 7 = 1$

$\chi = |V| - |E| + |F|$



$\chi = 8 - 8 = 0$

$\chi = |V| - |E| + |F|$



$\chi = 8 - 9 = -1$