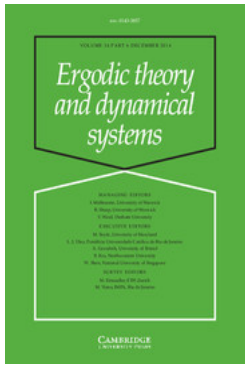

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Ergodic Theory and Dynamical Systems / *FirstView* Article / December 2014, pp 1 - 24
DOI: 10.1017/etds.2014.91, Published online: 12 December 2014

Link to this article: http://journals.cambridge.org/abstract_S0143385714000911

How to cite this article:
IONUT CHIFAN, THOMAS SINCLAIR and BOGDAN UDREA Inner amenability for groups and central sequences in factors. Ergodic Theory and Dynamical Systems, Available on CJO 2014 doi:10.1017/etds.2014.91

Ergod. Th. & Dynam. Sys. (First published online 2014), page 1 of 24*
doi:10.1017/etds.2014.91
*Provisional—final page numbers to be inserted when paper edition is published

Inner amenability for groups and central sequences in factors

IONUT CHIFAN†‡, THOMAS SINCLAIR§ and BOGDAN UDREA‡¶

† *Department of Mathematics, University of Iowa, 14 MacLean Hall, IA 52242, USA*
‡ *IMAR, Bucharest, Romania*
(e-mail: ionut-chifan@uiowa.edu)
§ *Department of Mathematics, University of California, Los Angeles, Box 951555, Los Angeles, CA 90095-1555, USA*
(e-mail: thomas.sinclair@math.ucla.edu)
¶ *Department of Mathematics, University of Illinois, Urbana Champaign, IL 61801, USA*
(e-mail: budrea@illinois.edu)

(*Received* 4 *February* 2014 *and accepted in revised form* 12 *August* 2014)

Abstract. We show that a large class of i.c.c., countable, discrete groups satisfying a weak negative curvature condition are not inner amenable. By recent work of Hull and Osin [Groups with hyperbolically embedded subgroups. *Algebr. Geom. Topol.* **13** (2013), 2635–2665], our result recovers that mapping class groups and $\mathrm{Out}(\mathbb{F}_n)$ are not inner amenable. We also show that the group-measure space constructions associated to free, strongly ergodic p.m.p. actions of such groups do not have property Gamma of Murray and von Neumann [On rings of operators IV. *Ann. of Math.* (2) **44** (1943), 716–808].

Contents

1	Introduction	1
2	Background and methods	5
3	Technical results	9
4	Central sequences, asymptotic relative commutants, and inner amenability	13
5	Further remarks	21
Acknowledgements		22
References		23

1. *Introduction*

The study of central sequences has occupied a prominent place in the classification of II_1 factors. In their seminal investigations Murray and von Neumann [**23**] defined a II_1 factor M to have *property Gamma* if there exists a net of unitaries (u_n) in M with

$\tau(u_n) \equiv 0$ and such that $\|xu_n - u_n x\|_2 \to 0$, for all $x \in M$. In particular, they showed that the free group factor $L(\mathbb{F}_2)$ does not have property Gamma (therefore, not hyperfinite), providing the first demonstration of non-isomorphic II_1 factors. The study of property Gamma subsequently played an important role in the celebrated classification results of McDuff [**19**] and Connes [**7**].

In the 1970s Effros [**9**] introduced an analog of property Gamma for discrete groups, which he termed *inner amenability*. A group Γ is called inner amenable if there exists a finite additive measure μ on the subsets $\Gamma\backslash\{e\}$ of total mass one such that $\mu(X) = \mu(\gamma^{-1}X\gamma)$ for all $X \subset \Gamma\backslash\{e\}$. Equivalently, Γ is inner amenable if there exists a net $\xi_n \in \ell^2(\Gamma\backslash\{e\})$ of unit-norm vectors such that $\|u_\gamma \xi_n - \xi_n u_\gamma\|_2 \to 0$ for all $\gamma \in \Gamma$. A trivial consequence of [**7**, Theorem 2.1] is that, for i.c.c. discrete groups (that is, for discrete groups for which every non-identity conjugacy class is infinite), inner amenability is a weaker property than the group von Neumann algebra having property Gamma. However, examples of inner amenable groups whose von Neumann algebras do not possess property Gamma have only been recently constructed by Vaes [**45**].

For II_1 factors without property Gamma, strong classification results have become achievable in a large part through the development of Popa's deformation/rigidity theory [**34–36**]. As the theory developed, it was readily noticed that fairly mild 'deformability' and 'rigidity' assumptions could be used to demonstrate the absence of property Gamma, cf. [**5**, **15**, **27**, **30**, **31**, **36**]. In parallel, it was noticed that modest 'negative curvature' assumptions on a discrete group could be used to show non-inner amenability [**8**, **13**]. The goal of this paper is to fully develop the connections between these results through deriving non-inner amenability of large classes of countable, discrete groups through operator algebraic methods, specifically the theory of 'weak' deformations developed in [**5**, **6**, **41**].

1.1. *Statement of results.* This paper is a continuation in the series of papers [**5**, **6**] exploring the consequences of negative-curvature phenomena in geometric group theory for the structure of group and group-measure space factors. The first in the series [**5**] dealt with structural results in the context of the strongest type of negative curvature condition, namely Gromov hyperbolicity. The essential result obtained therein was the extension of the strong solidity results of Ozawa and Popa [**29**, **30**] and of the second author [**41**] from lattices in rank-one semisimple Lie groups to Gromov hyperbolic groups in general. In the second paper in the series [**6**], this result was further refined in two ways: first, to cover all weakly amenable groups satisfying a weaker negative curvature condition, relative hyperbolicity; and second, to cover products of such groups. The starting point of all of these results is the conversion of the negative curvature condition into a natural cohomological-type condition (relative $\mathcal{QH}_{\mathrm{reg}}$/bi-exactness) which is used to construct a weak deformation of the von Neumann algebra to which generalized 'spectral gap rigidity' arguments of the type developed by Popa [**36**] and Ozawa and Popa [**29**, **30**] are applied to obtain the desired classification results. The results of [**5**, **6**] were subsequently extended to general crossed product factors of hyperbolic groups in [**39**].

Recent progress in geometric group theory has been on obtaining structural results for groups satisfying the much weaker negative curvature condition of 'acylindrical hyperbolicity,' cf. [**26**], or various equivalent geometric properties such as the condition of admitting a hyperbolic, WPD element, cf. [**3**], admitting a proper, infinite hyperbolically embedded subgroup [**8**], and weak acylindricity [**12**], among others. In particular, it was shown by Dahmani *et al* [**8**], that any group satisfying one of these conditions is not inner amenable. In parallel with these advances in geometric group theory, we introduce a cohomological-type version of weak negative curvature which we will use to classify the structure of central sequences for the group von Neumann algebra and the related group-measure space constructions.

Notation 1.1. Let Γ be a countable discrete group, and let $\mathcal{G}$ be a family of subgroups of Γ. In order to simplify the statements of the results, throughout the paper we will use the notation set forth here. We will say that Γ satisfies the condition NC relative to the family $\mathcal{G}$ (abbreviated $\mathsf{NC}(\mathcal{G})$) if Γ satisfies one of the following statements:

- Γ admits an unbounded quasi-cocycle into a non-amenable orthogonal representation which is mixing with respect to $\mathcal{G}$;
- Γ admits a symmetric array into a non-amenable orthogonal representation so that the array is proper with respect to $\mathcal{G}$.

The group Γ satisfies condition NC if it satisfies condition NC relative to the family consisting of the trivial subgroup.

The condition NC is satisfied for all groups in the class $\mathcal{D}_{\mathrm{reg}}$ of Thom [**43**], in particular all acylindrically hyperbolic groups, as well as the class $\mathcal{QH}$ of the first two authors [**5**]. As we point out below, the class NC also contains all groups with positive first ℓ^2-Betti number, which as a class were not previously known to be non-inner amenable. We refer the reader to §2 below for relevant terminology and examples.

In this paper we obtain a complete classification of the asymptotic central sequences of arbitrary crossed products factors $M = A \rtimes \Gamma$ associated with groups Γ satisfying condition $\mathsf{NC}(\mathcal{G})$. In more colloquial terms, we will be showing that all sequences which asymptotically commute with the entire factor M must asymptotically 'live' close to the (canonical) von Neumann subalgebras of M arising from the subgroups of $\mathcal{G}$. Basic examples can be constructed to show that this control is actually sharp. In particular, this result provides large natural classes of examples of i.c.c. groups whose factors do not possess property Gamma of Murray and von Neumann. We now state the results.

THEOREM A. *Let Γ be a countable discrete group together with a family of subgroups $\mathcal{G}$. Let $\Gamma \curvearrowright A$ be any trace preserving action on a finite von Neumann algebra and denote $M = A \rtimes \Gamma$. Also assume that ω is a free ultrafilter on the positive integers $\mathbb{N}$.*

If Γ satisfies condition $\mathsf{NC}(\mathcal{G})$, then for any sequence $(x_n)_n \in M' \cap M^\omega$ there exists a finite subset $\mathcal{F} \subseteq \mathcal{G}$ such that $(x_n)_n \in \vee_{\Sigma \in \mathcal{F}} (A \rtimes \Sigma)^\omega \vee M$.

COROLLARY B. *If Γ is an i.c.c., countable, discrete group which admits a non-degenerate, hyperbolically embedded subgroup, then $L\Gamma$ does not have property Gamma. If $\Gamma \curvearrowright (X, \mu)$ is any strongly ergodic probability measure preserving (p.m.p.) action, then the group-measure space factor $L^\infty(X) \rtimes \Gamma$ does not have property Gamma. In particular, this applies to non-virtually abelian mapping class groups $\mathcal{MCG}(\Sigma)$ for Σ a (punctured) closed, orientable surface as well as* $\mathrm{Out}(\mathbb{F}_n)$, *$n \geq 3$.*

For i.c.c. groups, the result may be sharpened to further rule out inner amenability. We mention in passing that while this result is group theoretical in nature, the proof is operator algebraic, rooted in Popa's deformation/rigidity theory.

THEOREM C. *Let Γ be an i.c.c. group together with a family of subgroups $\mathcal{G}$. Assume that Γ is i.c.c. over every subgroup $\Sigma \in \mathcal{G}$ (cf. Definition 4.3). If Γ satisfies condition* $\mathsf{NC}(\mathcal{G})$*, then Γ is not inner amenable.*

Since a non-amenable group is known to have positive first ℓ^2-Betti number if and only if it admits an unbounded 1-cocycle into its left-regular representation [**33**, Corollary 2.4], we have the following easy corollary. Surprisingly, to the best of our knowledge this is the first time this result has appeared in print, though we were informed by Taka Ozawa that he had previously obtained this result in unpublished work.

COROLLARY D. *If Γ is an i.c.c. countable discrete group with positive first ℓ^2-Betti number, then Γ is not inner amenable.*

When combined with the main result of Hull and Osin from [**16**] our theorem also recovers the following earlier result due to Dahmani *et al*.

COROLLARY E. (Dahmani *et al* [**8**]) *If Γ is an i.c.c., countable, discrete group which admits a non-degenerate, hyperbolically embedded subgroup, then Γ is not inner amenable.*

Additionally, for such groups the authors were also able to demonstrate in [**8**] simplicity of the reduced C^*-algebra $C^*_r(\Gamma)$. We were not able to obtain any positive results for the more general class of groups satisfying condition NC, though we remark on some possible connections between the results outlined here and C^*-simplicity in §5.2.

Recall, a II_1 factor is said to be prime if it is not isomorphic to a tensor product of diffuse factors.

QUESTION 1.2. *If Γ is an exact, non-amenable, i.c.c., countable, discrete group which admits an unbounded quasi-cocycle into $\ell^2(\Gamma)^{\oplus\infty}$, is $L\Gamma$ prime?*

Note that Peterson showed [**31**, Corollary 4.6] that primeness of $L\Gamma$ does follow from the much more restrictive assumption that Γ admits an unbounded 1-cocycle into $\ell^2(\Gamma)^{\oplus\infty}$ (the assumption of exactness is not necessary). Primeness is also known when the quasi-cocycle is proper, even extending to the case of proper arrays, by [**5**, Theorem A], though this is implicitly due to Ozawa ([**27**, Theorem 1] via [**5**, Remark 1.10]). An affirmative answer to the question would be sharp: by [**5**, Proposition 1.4] the group $\mathbb{F}_2 \times \mathbb{F}_2$, for instance, admits an unbounded (but not proper) array into its left-regular representation.

1.2. *Outline of the paper.* The next section contains the necessary background material, definitions, and examples. The third section consists of the statement and proofs of the main new technical results on quasi-cocycles. The proofs of the main results stated above, as well as other applications of the technique, form the fourth and last section of the paper.

2. *Background and methods*

2.1. *Arrays and quasi-cocycles.* Arrays were introduced by the first two authors in [**5**] as a language for unifying the concepts of length functions and 1-cocycles into orthogonal representations. In practice arrays can be used either to strengthen the concept of a length function by introducing a representation or to introduce some geometric flexibility to the concept of a 1-cocycle. See [**5**, §1] for an in-depth discussion of this concept and its relation with the phenomenon of negative curvature in geometric group theory.

Definition 2.1. Assume that Γ is a countable, discrete group together with $\mathcal{G} = \{\Sigma_i : i \in I\}$, a family of subgroups of Γ, and $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, an orthogonal representation. Following [**5**, Definition 1.4], we say that the group Γ admits an array into $\mathcal{H}$ if there exists a map $r : \Gamma \to \mathcal{H}$ which satisfies the following bounded equivariance condition:

$$\sup_{\delta\in\Gamma} \|r(\gamma\delta) - \pi_\gamma(r(\delta))\| = C(\gamma) < \infty \quad \text{for all } \gamma \in \Gamma.$$

An array r is said to be *symmetric* {*anti-symmetric*} if we have that

$$\pi_\gamma(r(\gamma^{-1})) = r(\gamma)\ \{\pi_\gamma(r(\gamma^{-1})) = -r(\gamma)\}$$

for all $\gamma \in \Gamma$. It is *proper relative to* $\mathcal{G}$ if for every $C > 0$ there are finite subsets $F \subset G$ and $\mathcal{K} \subset \mathcal{G}$ such that

$$B_C := \{\gamma \in \Gamma : \|r(\gamma)\| \le C\} \subset \bigcup_{K\in\mathcal{K}} FKF.$$

For a detailed list of properties of groups that admit non-trivial arrays the reader may consult [**5**, **6**].

The main examples of arrays on groups are quasi-cocycles. As before let Γ be a countable group together with $\mathcal{G} = \{\Sigma_i : i \in I\}$ a family of subgroups of Γ and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation.

Definition 2.2. A map $q : \Gamma \to \mathcal{H}$ is called a quasi-cocycle if there exists a constant $D \ge 0$ such that

$$\|q(\gamma\lambda) - \pi_\gamma(q(\lambda)) - q(\gamma)\| \le D \quad \text{for all } \gamma, \lambda \in \Gamma. \tag{2.1}$$

The infimum over all such D is denoted by $D(q)$ and is called the defect of q. When the defect is zero q is actually a 1-cocycle with coefficients in π (i.e. an element in $Z^1(\Gamma, \pi)$). Any bounded map $b : \Gamma \to \mathcal{H}$ is automatically a quasi-cocycle whose error does not exceed three times the uniform bound of b.

It was observed by Thom [**43**] that any quasi-cocycle lies within bounded distance from an anti-symmetric one. We denote the space of anti-symmetric quasi-cocycles associated to the representation π as $QZ^1_{as}(\Gamma, \pi)$ and the subspace of those which are bounded as $QB^1_{as}(\Gamma, \pi)$. The first quasi-cohomology space is then defined to be $QH^1_{as}(\Gamma, \pi) := QZ^1_{as}(\Gamma, \pi)/QB^1_{as}(\Gamma, \pi)$. In particular, if π is the left-regular representation λ_Γ, then $QH^1_{as}(\Gamma, \lambda_\Gamma)$ has the structure of a right $L\Gamma$-module.

Definition 2.3. A group Γ is said to be in the class $\mathcal{D}_{\mathrm{reg}}$ if $\dim_{L\Gamma} QH^1_{as}(\Gamma, \lambda_\Gamma) \neq 0$.

By [**43**, Lemma 2.8] and [**33**, Corollary 2.4] it is observed that a countable, discrete group Γ is in the class $\mathcal{D}_{\mathrm{reg}}$ if and only if it admits an unbounded quasi-cocycle into its left-regular representation.

2.2. *Groups satisfying condition* NC. We will now describe some specific examples and constructions of classes of groups satisfying condition NC with respect to some explicit families of subgroups. While all groups which are (relatively) bi-exact belong to this class, stronger results, cf. [**4**–**6**, **28**], are known in this case, so we will focus our attention here on weaker 'negative curvature' conditions which can be used to construct unbounded quasi-cocycles. Before doing so, in order to make the exposition more self-contained and to furnish some familiar examples, we will recall how to construct quasi-cocycles algebraically for classes of groups arising from canonical constructions like semi-direct products, amalgamated free products, and HNN-extensions. For more details we refer to [**6**, §2.2].

Examples 2.4. Each group in the following classes satisfies condition NC with respect to the associated family of subgroups $\mathcal{G}$.

(a) For $\Sigma < \Gamma_1, \Gamma_2$ groups, denote by $\Gamma := \Gamma_1 \star_\Sigma \Gamma_2$ the corresponding amalgamated free product and assume that Σ is not co-amenable in Γ. Then it is well known that $H^1(\Gamma, \ell^2(\Gamma/\Sigma)) \neq \{0\}$ and we consider $\mathcal{G} := \{\Sigma\}$.

(b) For $\Sigma < \Gamma$ groups and $\theta : \Sigma \to \Gamma$ a monomorphism, denote by $\Gamma := \mathrm{HNN}(\Gamma, \Sigma, \theta)$ the corresponding HNN-extension and assume that Σ is not co-amenable in Γ. Then again we have $H^1(\Gamma, \ell^2(\Gamma/\Sigma)) \neq \{0\}$ and we let $\mathcal{G} := \{\Sigma\}$.

(c) Let Γ be a non-amenable group that acts on a tree $\mathcal{T} = (\mathsf{V}, \mathsf{E})$ and for each edge $e \in \mathsf{E}$ denote by $\Gamma_e := \{\gamma \in \Gamma : \gamma e = e\}$ its stabilizer group. Since Γ acts on a tree, there exists a 1-cocycle into the semi-regular orthogonal representation $\lambda_{\mathsf{E}} := \bigoplus_{e\in\mathsf{E}} \ell^2(\Gamma/\Gamma_e)$ where the group acts by left translation on each summand. If we assume the trivial representation 1_Γ is not weakly contained in λ_{E} (e.g. when all Γ_e are amenable) then it is clear that λ_{E} is non-amenable and mixing with respect to the family $\mathcal{G} := \{\Gamma_e : e \in \mathsf{E}\}$.

PROPOSITION 2.5. *Let* H, Γ *be non-trivial countably discrete groups, let* I *be an infinite* Γ*-set, and consider the generalized wreath product group* $H \wr_I \Gamma := H^{(I)} \rtimes_\sigma \Gamma$*; here we have denoted* $H^{(I)} := \bigoplus_I H$ *and by* σ *the natural shift action of* Γ *on* $H^{(I)}$ *induced by the action of* Γ *on* I*. For every* $K \subset I$*, denote by* $\Gamma_K := \{\gamma \in \Gamma : \gamma K = K\}$ *the stabilizing group of the subset* K *and let* $\mathcal{G} := \{H^{(I)} \rtimes_\sigma \Gamma_i : i \in I\}$*. If* H *is non-amenable then* $H \wr_I \Gamma$ *satisfies condition* $\mathsf{NC}(\mathcal{G})$*. Also, if* H *is amenable,* Γ *is non-amenable, and all the stabilizers* Γ_i *are amenable for all* $i \in I$ *then again* $H \wr_I \Gamma$ *satisfies condition* $\mathsf{NC}(\mathcal{G})$.

Proof. Consider the canonical orthogonal representation $\pi : H^{(I)} \to \mathcal{O}(\oplus_I \ell^2(H))$, and let $\pi' : \Gamma \to \mathcal{O}(\bigoplus_I \ell^2(H))$ be defined as $\pi'_\gamma(\bigoplus_i \xi_i) = \bigoplus_i \xi_{\gamma^{-1}i}$ for all $\gamma \in \Gamma$. One can check that π' is also an orthogonal representation satisfying $\pi'_\gamma \circ \pi_h \circ \pi'_{\gamma^{-1}} = \pi_{\sigma_\gamma(h)}$ for all $h \in H^{(I)}, \gamma \in \Gamma$. Hence $\tilde{\pi} : H \wr_I \Gamma \to \mathcal{O}(\bigoplus_I \ell^2(H))$ defined by $\tilde{\pi}_{h\gamma} = \pi_h \circ \pi'_\gamma$ for all $h \in H^{(I)}$ and $\gamma \in \Gamma$, is also an orthogonal representation which extends π and π', respectively.

We observe that the map $c : H^{(I)} \to \bigoplus_I \ell^2(H)$ given by $c(h) = c((h_i)_i) = \bigoplus_i (\delta_e - \delta_{h_i})$ defines a 1-cocycle which satisfies the following compatibility relation: $c(\sigma_\gamma(h)) = \tilde{\pi}_\gamma(c(h))$ for all $h \in H^{(I)}$ and $\gamma \in \Gamma$. This enables one to define a 1-cocycle $\tilde{c} : H \wr_I \Gamma \to \bigoplus_I \ell^2(H)$ by letting $\tilde{c}(h\gamma) = c(h)$ for all $h \in H^{(I)}$ and $\gamma \in \Gamma$. A straightforward calculation shows that $\|\tilde{c}(h\gamma)\|^2 = 2\#(\mathrm{supp}(h))$ for all $h \in H^{(I)}$ and $\gamma \in \Gamma$, where $\mathrm{supp}(h) \subset I$ denotes the support of h. Since I is infinite it follows that $\tilde{c}$ is unbounded; hence, $H^1(H \wr_I \Gamma, \bigoplus_I \ell^2(H)) \neq 0$. Notice that when H is non-amenable we have that π is non-amenable, whence $\tilde{\pi}$ is non-amenable without any additional assumption on Γ. When H is amenable a basic calculation shows that π' is unitarily equivalent to $\bigoplus_F \ell^2(\Gamma/\Gamma_F)$, where the direct sum is over some finite subsets $F \subset I$. Notice that $\Gamma_F = \bigcap_{\text{some } j \in F} \Gamma_{O_j}$ where O_j are the (finite) orbits of the natural action of G_F on F. Moreover, for every such j there exists $i_j \in I$ such that $\Gamma_{i_j} < \Gamma_{O_j}$ is a subgroup of index $\#(O_j)$. This further implies that π' is weakly contained in $\bigoplus_{S \subset I,\ \text{finite}} \ell^2(\Gamma/(\bigcap_{i \in S} \Gamma_i))$; here the direct sum is over all finite subsets of I. Hence, if we assume that the trivial representation 1_Γ is not weakly contained in $\bigoplus_{S \subset I,\ \text{finite}} \ell^2(\Gamma/(\bigcap_{i \in S} \Gamma_i))$ (e.g. when Γ is non-amenable and all stabilizers Γ_i are amenable) then we conclude that π', whence $\tilde{\pi}$, is non-amenable.

Finally, we briefly check that $\tilde{\pi}$ is mixing with respect to $\mathcal{G}$. For this fix $\varepsilon > 0$ and $\xi, \eta \in \bigoplus_I \ell^2(H)$. We can assume without any loss of generality that the supports $\mathrm{supp}(\xi) = F_1$, $\mathrm{supp}(\eta) = F_2$ are finite subsets of I. Using the definitions notice that there exist finite subsets $K \subset \Gamma$, $L \subset I$ such that $\{\gamma \in \Gamma : \gamma F_1 \cap F_2 \neq \emptyset\} \subseteq K(\bigcup_{i \in L} \Gamma_i)K$. Thus, for every $h\gamma \in H \wr_I \Gamma \backslash K(\bigcup_{i \in L} H^{(I)} \rtimes_\sigma \Gamma_i)K$ we have that

$$\langle \tilde{\pi}_{hg}(\xi), \eta \rangle = \langle \tilde{\pi}_g(\xi), \pi_{h^{-1}}(\eta) \rangle = \sum_i \langle \xi_{\gamma^{-1} i}, \pi_{h_i^{-1}}(\eta_i) \rangle$$
$$= \sum_{i \in F_2 \cap \gamma F_1} \langle \xi_{\gamma^{-1} i}, \pi_{h_i^{-1}}(\eta_i) \rangle = 0 < \varepsilon,$$

as desired. □

Another class of examples comes from lattices in locally compact, second countable (l.c.s.c.) groups.

Example 2.6. Consider a l.c.s.c. group $G = G_1 \times G_2$, where G_1 has property (HH) of Ozawa and Popa [**30**], i.e. G_1 admits a proper cocycle into a non-amenable, mixing orthogonal representation. If $\Gamma < G$ is a lattice then Γ satisfies condition $\mathsf{NC}(\mathcal{K})$ for $\mathcal{K}$ the set of subgroups $K < \Gamma$ such that the projection $pr_1(K)$ of K into G_1 is precompact. (Note that the lattice assumption is only necessary to ensure non-amenability of the restricted representation.)

We will now describe some recent, innovative methods for building quasi-cocycles through the use of geometric methods in group theory. Some of the first results in this direction come from the seminal work of Mineyev [**20**] and Mineyev *et al* [**21**], who showed that if Γ is a Gromov hyperbolic group, then Γ admits a proper quasi-cocycle into a finite multiple of its left-regular representation, whence Γ belongs to $\mathcal{D}_{\text{reg}}$. This was generalized to groups which are relatively hyperbolic to a family of subgroups by Mineyev and Yaman [**22**]. Hamenstädt [**12**] showed that all weakly acylindrical groups—in particular, non-virtually abelian mapping class groups and $\mathrm{Out}(\mathbb{F}_n)$, $n \geq 2$—belong to the class $\mathcal{D}_{\text{reg}}$. A unified approach to these results was recently developed by Hull and Osin [**16**] and independently by Bestvina *et al* [**3**]. Specifically, they were able to show that every group Γ which admits a non-degenerate, hyperbolically embedded subgroup belongs to the class $\mathcal{D}_{\text{reg}}$ via an extension theorem for quasi-cohomology. In fact, by very recent work of Osin [**26**] the weak curvature conditions used in both papers, as well as Hamenstädt's weak acylindricity condition, are all equivalent to the notion of 'acylindrical hyperbolicity' formulated by Bowditch, cf. [**26**].

Examples 2.7. Collecting these results together, the following families of groups are known to be acylindrically hyperbolic. In particular they belong to the class $\mathcal{D}_{\text{reg}}$, thus satisfy condition NC:
(a) Gromov hyperbolic groups [**20**, **21**];
(b) groups which are hyperbolic relative to a family of subgroups as in [**22**];
(c) the mapping class group $\mathcal{MCG}(\Sigma)$ for any (punctured) closed, orientable surface Σ, provided that it is not virtually abelian [**12**];
(d) $\text{Out}(\mathbb{F}_n)$, $n \geq 2$ [**12**];
(e) groups which admit a proper isometric action on a proper CAT(0) space [**42**].

We remark that it is unclear whether condition NC is closed under finite direct sums, though the following partial stability result is easily observed; see [**5**, Proposition 1.7].

PROPOSITION 2.8. *Let Γ_1 and Γ_2 be countable, discrete groups, and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be respective families of subgroups. Consider the direct product $\Gamma = \Gamma_1 \times \Gamma_2$ equipped with the family of subgroups $\mathcal{G} := \{\Sigma_1 \times \Gamma_2 : \Sigma_1 \in \mathcal{G}_1\} \cup \{\Gamma_1 \times \Sigma_2 : \Sigma_2 \in \mathcal{G}_2\}$. If both Γ_1 and Γ_2 either admit a symmetric array into a non-amenable representation which is proper with respect to $\mathcal{G}_i$ or admit an unbounded quasi-cocycle into a non-amenable representation which is mixing with respect to $\mathcal{G}_i$, $i = 1, 2$, then the same holds for Γ with respect to $\mathcal{G}$.*

On the other hand, it is known that the classes of non-Gamma factors and non-inner amenable groups are closed under, respectively, finite tensor products (a non-trivial result of Connes [**7**]) and finite direct sums Bekka [**2**, Theorem 2.4]. Precisely, Γ is not inner amenable if and only if the orthogonal representation induced by action of Γ by conjugation on $\ell^2(\Gamma \backslash \{e\})$ is non-amenable. This suggests that the condition NC as formulated may not be an optimal condition for establishing results along the lines of those stated in the introduction.

2.3. *A family of deformations arising from arrays.* We briefly recall from [**5**] the construction of a deformation from an array based on the main construction in [**41**]. To do this we need to first recall the construction of the Gaussian action associated to a representation (see for example [**32**]). Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation on a real Hilbert space. Then there exists a standard probability space (X^π, μ) so that the abelian von Neumann algebra $L^\infty(X^\pi, \mu)$ is generated by a family of unitaries $\omega(\xi)$, $\xi \in \mathcal{H}$, subject to the following relations:
(1) $\omega(\xi_1)\omega(\xi_2) = \omega(\xi_1 + \xi_2)$ for any $\xi_1, \xi_2 \in \mathcal{H}$;
(2) $\omega(-\xi) = \omega(\xi)^*$ for any $\xi \in \mathcal{H}$; and
(3) $\int \omega(\xi)\, d\mu = \exp(-\|\xi\|^2)$, for any $\xi \in \mathcal{H}$.

The *Gaussian action* of Γ on (X^π, μ) is defined via the action on $L^\infty(X^\pi)$ by $\tilde{\pi}_\gamma(\omega(\xi)) = \omega(\pi_\gamma(\xi))$ for all $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$.

Definition 2.9. Let $\Gamma \curvearrowright^\sigma (A, \tau)$ be a trace-preserving action of Γ on a finite von Neumann algebra A and let $M = A \rtimes_\sigma \Gamma$ be the cross-product von Neumann algebra. The *Gaussian dilation* associated to M is the von Neumann algebra $\tilde{M} = (A \bar{\otimes} L^\infty(X^\pi)) \rtimes_{\sigma\otimes\tilde{\pi}} \Gamma$.

Let $q : \Gamma \to \mathcal{H}$ be an array for the representation π as above. The deformation is constructed as follows. For each $t \in \mathbb{R}$, define the unitary $V_t \in \mathcal{U}(L^2(A) \otimes L^2(X^\pi) \otimes \ell^2(\Gamma))$ by

$$V_t(a \otimes d \otimes \delta_\gamma) := a \otimes \omega(tq(\gamma))d \otimes \delta_\gamma$$

for all $a \in L^2(A)$, $d \in L^2(X^\pi)$, and $\gamma \in \Gamma$. In [**5**] it was proved that V_t is a strongly continuous one parameter group of unitaries having the following transversality property.

PROPOSITION 2.10. [**5**, Lemma 2.8] *For each t and any $\xi \in L^2(M)$, we have*

$$2\|V_t(\xi) - e \cdot V_t(\xi)\|_2^2 \geq \|\xi - V_t(\xi)\|_2^2, \tag{2.2}$$

where e denotes the orthogonal projection of $L^2(\tilde{M})$ onto $L^2(M)$.

The following 'asymptotic bimodularity' property of the deformation V_t is the most crucial consequence of the array property. The following proposition is essentially [**5**, Lemma 2.6].

PROPOSITION 2.11. *Let Γ be a group, let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation and let $q : \Gamma \to \mathcal{H}$ be an array for π. Assume that $\Gamma \curvearrowright^\sigma A$ is a trace preserving action and let $\Gamma \curvearrowright^{\sigma\otimes\pi} A\bar{\otimes}L^\infty(X^\pi)$ be the Gaussian construction associated to π. Denote by $M = A \rtimes \Gamma$ and $\tilde{M} = (A\bar{\otimes}L^\infty(X^\pi)) \rtimes_{\sigma\otimes\tilde{\pi}} \Gamma$ the corresponding crossed product von Neumann algebras so that $M \subset \tilde{M}$. To q we associate the path of isometries $V_t : L^2(M) \to L^2(\tilde{M})$ obtained by restricting the V_t's constructed above. Then for every $x, y \in A \rtimes_{\sigma,r} \Gamma$ in the reduced C^*-crossed product subalgebra of $A \rtimes \Gamma$ we have*

$$\lim_{t\to 0} \sup_{\|\xi\|_2 \leq 1} \|xV_t(\xi)y - V_t(x\xi y)\|_2 = 0. \tag{2.3}$$

Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of Γ which admits no non-zero invariant vectors, i.e. ρ is ergodic. In this case ρ has *spectral gap* if and only if it does not weakly contain the trivial representation. The orthogonal representation ρ is *non-amenable* if $\rho \otimes \rho$ is ergodic and has spectral gap.

PROPOSITION 2.12. [**32**, Proposition 2.7] *The orthogonal representation ρ is non-amenable if and only if the Koopman representation $\tilde{\rho} : \Gamma \to \mathcal{U}(L^2(X^\rho) \ominus \mathbb{C}1)$ associated to the Gaussian action is ergodic and has spectral gap.*

3. *Technical results*

In this section we present the new technique for working with quasi-cocycles which will allow us to prove the main results of the paper. Studying the properties of groups through various Hilbert space embeddings, e.g. quasi-cocycles, which are compatible with some representation has emerged as a fairly important tool which captures many interesting aspects regarding the internal (algebraic) structure of the group. The principle is parallel to the use of geometric techniques (via a word-length metric) to deduce algebraic structure—the difference being that while geometric techniques focus on deducing structure from assumptions on the length function itself, quasi-cocycle (or array) techniques impose structure from the existence of a generic Hilbert space-valued 'length' function which is compatible with a specific representation.

Continuing this trend we show next that if the representation is mixing then the finite radius balls with respect to the natural length function induced by the quasi-cocycle are, in a sense, highly malnormal. As a consequence we show that we have large sets in the group which are asymptotically free.

THEOREM 3.1. *Let Γ be a countable, discrete group, $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation which is mixing with respect to some family $\mathcal{G}$, and $q : \Gamma \to \mathcal{H}$ be a quasi-cocycle with defect D. For every $C \geq 0$ we denote $B_C := \{\gamma \in \Gamma \,:\, \|q(\gamma)\| \leq C\}$. For every finite set $F \subset \Gamma \setminus B_{2C+2D}$ there exists a subset $K \subset B_C$ which is small with respect to the family $\mathcal{G}$ such that*

$$F(B_C \setminus K) \cap (B_C \setminus K)F = \emptyset.$$

Moreover, if the quasi-cocycle q is bounded on each group in $\mathcal{G}$, then for every finite set $F \subset \Gamma \setminus B_{6C+6D}$ one can find a subset $K \subset B_C$ which is small with respect to the family $\mathcal{G}$ and satisfies

$$F(B_C \setminus (K^2 \cup K))F(B_C \setminus (K^2 \cup K)) \cap (B_C \setminus (K^2 \cup K))F(B_C \setminus (K^2 \cup K))F = \emptyset.$$

The proof rests on the following key technical result.

PROPOSITION 3.2. *Let Γ be a countable group together with a family of subgroups $\mathcal{G}$, $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation that is mixing with respect to $\mathcal{G}$, and $q : \Gamma \to \mathcal{H}$ be a quasi-cocycle. For every $C \geq 0$ we denote*

$$B_C := \{\gamma \in \Gamma : \|q(\gamma)\| \leq C\}.$$

Fix $C \geq 0$ and an integer $\ell \geq 2$. Also let $k_1, k_2, \ldots, k_\ell \in \Gamma$ be elements such that, for each $1 \leq i \leq \ell$ there exists an infinite sequence $(\gamma_{n,i})_n \in B_C$ such that for all n we have $k_1\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3} \cdots k_\ell\gamma_{n,\ell} = e$ and each of the following sets $\{\gamma_{n,1} : n \in \mathbb{N}\}$, $\{\gamma_{n,1}k_2\gamma_{n,2} : n \in \mathbb{N}\}, \{\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3} : n \in \mathbb{N}\}, \ldots, \{\gamma_{n,1}k_2 \cdots \gamma_{n,\ell-2}k_{\ell-1}\gamma_{n,\ell-1} : n \in \mathbb{N}\}$ tends to infinity with respect to $\mathcal{G}$. Then $k_1 \in B_{(\ell-1)(2C+2D)}$.

Proof. To begin, we claim that for all $x_1, x_2, \ldots, x_m \in B_C$, $b_2, b_3, \ldots, b_m \in \Gamma$, and $k \in \mathcal{H}$ we have

$$\begin{aligned} |\langle q(x_1b_2x_2 \cdots x_{m-1}b_mx_m), k\rangle| &\leq (m-1)(2C+2D)\|k\| \\ &\quad + \sum_{i=2}^{m} |\langle \pi_{x_1b_2x_2\cdots b_{i-1}x_{i-1}}(q(b_i)), k\rangle|. \end{aligned} \tag{3.1}$$

To prove our statement note that since $k_1^{-1} = \gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3} \cdots k_\ell\gamma_{n,\ell}$, then (3.1) implies

$$\begin{aligned}\|q(k_1^{-1})\|^2 &= \langle q(\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3}\cdots k_\ell\gamma_{n,\ell}), q(k_1^{-1})\rangle\\ &\leq (\ell-1)(2C+2D)\|q(k_1^{-1})\|\\ &\quad+\sum_{i=2}^{\ell}|\langle\pi_{\gamma_{n,1}k_2\gamma_{n,2}\cdots k_{i-1}\gamma_{n,i-1}}(q(k_i)), q(k_1^{-1})\rangle|. \end{aligned}\tag{3.2}$$

Since each of the sets $\{\gamma_{n,1} : n\in\mathbb{N}\}$, $\{\gamma_{n,1}k_2\gamma_{n,2} : n\in\mathbb{N}\}$, ..., $\{\gamma_{n,1}k_2\gamma_{n,2}\cdots k_{\ell-1}\gamma_{n,\ell-1} : n\in\mathbb{N}\}$ tends to infinity relative to $\mathcal{G}$ and π is a mixing relative to $\mathcal{G}$, taking the limit as $n\to\infty$ in (3.2) we get

$$\|q(k_1^{-1})\|^2 \leq (\ell-1)(2C+2D)\|q(k_1^{-1})\|.$$

Hence, by anti-symmetry we have

$$\|q(k_1)\| = \|q(k_1^{-1})\| \leq (\ell-1)(2C+2D).$$

To prove the claim we argue by induction on m. When $m=2$, using the quasi-cocycle relation (2.1) and the Cauchy–Schwarz inequality we have that

$$\begin{aligned}|\langle q(x_1b_2x_2), k\rangle| &\leq D\|k\| + |\langle q(x_1)+\pi_{x_1}(q(b_2x_2)), k\rangle|\\ &\leq (D+\|q(x_1)\|)\|k\| + |\langle\pi_{x_1}(q(b_2x_2)), k\rangle|\\ &\leq (C+2D)\|k\| + |\langle\pi_{x_1}(q(b_2)), k\rangle| + |\langle\pi_{x_1b_2}(q(x_2)), k\rangle|\\ &\leq (2C+2D)\|k\| + |\langle\pi_{x_1}(q(b_2)), k\rangle|.\end{aligned}$$

For the inductive step assume (3.1) holds for $2, \ldots, m$; thus, we have that

$$\begin{aligned}&|\langle q(x_1b_2\cdots b_mx_mb_{m+1}x_{m+1}), k\rangle|\\ &\quad\leq |\langle\pi_{x_1b_2}q(x_2b_3\cdots b_mx_mb_{m+1}x_{m+1}), k\rangle| + (C+2D)\|k\| + |\langle\pi_{x_1}(q(b_2)), k\rangle|\\ &\quad\leq m(2C+2D)\|k\| + \sum_{i=2}^{m}|\langle\pi_{x_1b_2x_2\cdots b_{i-1}x_{i-1}}(q(b_i)), k\rangle|,\end{aligned}\tag{3.3}$$

which proves the claim. □

Proof of Theorem 3.1. Let $k_1, k_2\in F\subset\Gamma\backslash B_{2C+2D}$ be fixed elements and let $(\gamma_{n,1})_n$, $(\gamma_{n,2})_n$ be sequences in B_C such that $k_1B_C\cap B_Ck_2 = \{k_1\gamma_{n,1} = \gamma_{n,2}k_2 : n\in\mathbb{N}\}$. In particular we have $k_1\gamma_{n,1}k_2^{-1}\gamma_{n,2}^{-1} = e$ for all n, so by the previous proposition (for $\ell = 2$) the set $\{\gamma_{n,1}^{-1} : n\in\mathbb{N}\}$ is small over $\mathcal{G}$ and so is $K_{k_1,k_2} = \{\gamma_{n,1} : n\in\mathbb{N}\}$. Indeed, if the set were not small with respect to $\mathcal{G}$, one could find a subsequence tending to infinity with respect to $\mathcal{G}$, whence $k_1\in B_{2C+2D}$, a contradiction. This entails that $k_1(B_C\backslash K_{k_1,k_2})\cap(B_C\backslash K_{k_1,k_2})k_2 = \emptyset$ and hence if we let $K = \bigcup_{k_1,k_2\in F}K_{k_1,k_2}$ we see that K is small with respect to $\mathcal{G}$ and $F(B_C\backslash K)\cap(B_C\backslash K)F = \emptyset$.

For the second part let $k_1, k_2, k_3, k_4\in F\subset\Gamma\backslash B_{6C+6D}$ and assume that there exist infinite sequences $(\gamma_{n,i})_n\in B_C$ with $1\leq i\leq 4$ such that $k_1B_Ck_2B_C\cap B_Ck_3B_Ck_4 := \{k_1\gamma_{n,1}k_2\gamma_{n,2} = \gamma_{n,3}k_3\gamma_{n,4}k_4 : n\in\mathbb{N}\}$. This further implies that $k_1\gamma_{n,1}k_2\gamma_{n,2}k_4^{-1}\gamma_{n,4}^{-1}k_3^{-1}\gamma_{n,3} = e$, for all $n\in\mathbb{N}$. Then from the previous proposition it follows that at least one of the sets $\{\gamma_{n,1} : n\in\mathbb{N}\}$, $\{\gamma_{n,1}k_2\gamma_{n,2} : n\in\mathbb{N}\}$, or $\{\gamma_{n,1}k_2\gamma_{n,2}k_4^{-1}\gamma_{n,4}^{-1} : n\in\mathbb{N}\}$ must be small with respect to $\mathcal{G}$. If we have either the set $\{\gamma_{n,1} : n \in \mathbb{N}\}$ or the set $\{\gamma_{n,1}k_2\gamma_{n,2}k_4^{-1}\gamma_{n,4}^{-1} : n \in \mathbb{N}\} = \{k_1^{-1}\gamma_{n,3}^{-1}k_3 : n \in \mathbb{N}\}$ (and hence $\{\gamma_{n,3} : n \in \mathbb{N}\}$!) is small with respect to $\mathcal{G}$ then the conclusion follows immediately. So it remains to analyze the case when $\{\gamma_{n,1} : n \in \mathbb{N}\}$ tends to infinity with respect to $\mathcal{G}$ while the set $\{\gamma_{n,1}k_2\gamma_{n,2} : n \in \mathbb{N}\}$ is small with respect to $\mathcal{G}$. Here we only need to investigate the case when there exists an infinite sequence of positive integers $(r_n)_n$ such that $\{\gamma_{r_n,1} : n \in \mathbb{N}\}$ tends to infinity with respect to $\mathcal{G}$, elements $m_1, m_2 \in \Gamma$, and a sequence of elements $a_n \in \Sigma$ for some $\Sigma \in \mathcal{G}$ such that $\gamma_{r_n,1}k_2\gamma_{r_n,2} = m_1 a_n m_2$ for all $n \in \mathbb{N}$. Note that the latter equation can be rewritten as

$$k_2\gamma_{r_n,2}m_2^{-1}a_n^{-1}m_1^{-1}\gamma_{r_n,1} = e \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

If the set $\{\gamma_{r_n,2} : n \in \mathbb{N}\}$ would tend to infinity with respect to $\mathcal{G}$ then, as $\{k_2\gamma_{r_n,2}m_2^{-1}a_n^{-1} : n \in \mathbb{N}\} = \{\gamma_{r_n,1}^{-1}m_1 : n \in \mathbb{N}\}$, we would have by the previous proposition that $k_2 \in B_{4C+4D}$ which is a contradiction. Therefore $\{\gamma_{r_n,2} : n \in \mathbb{N}\}$ must be small with respect to $\mathcal{G}$, and by equation (3.4) it follows that there exists a set K which is small with respect to $\mathcal{G}$ such that $\{\gamma_{r_n,1} : n \in \mathbb{N}\} \subset K^2$. This gives the desired conclusion.

In the case of mixing representations we get the following sharper result.

COROLLARY 3.3. *Let Γ be a countable, discrete group, and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be a mixing orthogonal representation. Assume $q : \Gamma \to \mathcal{H}$ is a quasi-cocycle and for every $C \geq 0$ we denote $B_C := \{\gamma \in \Gamma \ : \ \|q(\gamma)\| \leq C\}$. Let $C \geq 0$ and let $k_1, k_2, \ldots, k_\ell \in \Gamma$ be elements such that, for each $1 \leq i \leq \ell$ there exists a sequence $(\gamma_{n,i})_n \in B_C$ which tends to infinity such that we have $k_1\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3}\cdots k_\ell\gamma_{n,\ell} = e$ for all n. Then there exists $1 \leq j \leq n$ such that $k_j \in B_{\ell(C+D)}$. In other words, for every finite set $F \subset \Gamma \backslash B_{2\ell(C+D)}$ there exists a finite subset $K \subset B_C$ such that*

$$e \notin [F(B_C\backslash K)]^\ell := \underbrace{[F(B_C\backslash K)][F(B_C\backslash K)]\cdots[F(B_C\backslash K)]}_{\ell\text{-times}}. \tag{3.5}$$

In particular, for every finite set $F \subset \Gamma\backslash B_{2\ell(C+D)}$ and every $1 \leq \kappa \leq \ell$ there exists a finite subset $K \subset B_C$ such that

$$[F(B_C\backslash K)]^\kappa \cap [(B_C\backslash K)F]^{\ell-\kappa} = \emptyset.$$

Proof. First we claim that there exist two positive integers $1 \leq s < t \leq \ell$, an infinite sequence $(r_n)_n$ of positive integers, and an element $k_t' \in \Gamma$ such that for all n we have

$$\begin{aligned} &|\{k_s\gamma_{r_n,s} : n \in \mathbb{N}\}| = |\{k_s\gamma_{r_n,s}k_{s+1}\gamma_{r_n,s+1} : n \in \mathbb{N}\}| \\ &\quad = \cdots = |\{k_s\gamma_{r_n,s}k_{s+1}\gamma_{r_n,s+1}\cdots k_{t-1}\gamma_{r_n,t-1} : n \in \mathbb{N}\}| = \infty; \end{aligned} \tag{3.6}$$

$$k_s\gamma_{r_n,s}k_{s+1}\gamma_{r_n,s+1}\cdots k_{t-1}\gamma_{r_n,t-1}k_t'\gamma_{r_n,t} = e. \tag{3.7}$$

Next we observe that if this is the case then applying Proposition 3.2 above it follows that $k_s \in B_{2(t-s)(C+D)} \subseteq B_{2\ell(C+D)}$. The remaining part of the statement follows easily from this.

Therefore, it only remains to show that the claimed equations (3.6) and (3.7) hold. We proceed by induction on ℓ. When $\ell = 2$ the statement is trivial. To prove the inductive step, assume the statements hold for all $2 \leq m \leq \ell - 1$ and we will show it for ℓ. Notice that since by assumption we have $k_1\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3}\cdots k_\ell\gamma_{n,\ell}=e$ for all n, then there exists a smallest integer $2\le d_1\le\ell$ such that the sets $\{k_1\gamma_{n,1}:n\in\mathbb{N}\}$, $\{k_1\gamma_{n,1}k_2\gamma_{n,2}:n\in\mathbb{N}\}$, $\ldots,\{k_1\gamma_{n,1}k_2\gamma_{n,2}\cdots k_{d_1-1}\gamma_{n,d_1-1}:n\in\mathbb{N}\}$ are infinite while $\{k_1\gamma_{n,1}k_2\gamma_{n,2}\cdots k_d\gamma_{n,d_1}:n\in\mathbb{N}\}$ is finite. If $d_1=\ell$ then (3.6) and (3.7) follow trivially.

If $d_1\le\ell-1$ there exists an infinite sequence $(a_n)_n$ of integers and $c\in\Gamma$ such that $k_1\gamma_{a_n,1}k_2\gamma_{a_n,2}\cdots k_{d_1}\gamma_{a_n,d_1}=c$, for all n, whence we have that $k_2\gamma_{a_n,2}\cdots k_{d_1}\gamma_{a_n,d_1}k_1'\gamma_{a_n,1}=e$, for all n, where we denoted $k_1'=c^{-1}k_1$. In this case (3.6) and (3.7) follow from the induction assumption. □

We note that the previous corollary can be easily generalized to the case of quasi-cocycles into representations π which are mixing relative to a family of subgroups $\mathcal{G}$. The statement is virtually the same with the exception that, rather than a finite set, in this case K will be a set which is small with respect to $\mathcal{G}$ and equation (3.5) will hold with K replaced by $\bigcup_{j=1}^{[\ell/2]+1}K^j$. The proof is also very similar to the one presented above and we leave it to the reader.

Remark 3.4. The above results are an approximate translation to the quasi-cocycle perspective of [**8**, Proposition 2.8] which states that if $\Lambda<\Gamma$ hyperbolically embedded, then Λ is *almost malnormal* in Γ—i.e. $|\Lambda\cap\gamma\Lambda\gamma^{-1}|<\infty$ whenever $\gamma\notin\Lambda$. Indeed [**16**, Theorem 4.2] seems to suggest (though it is not explicitly proven) that starting from $0\in QZ^1_{as}(\Lambda,\lambda_\Lambda^{\oplus\infty})$, one can construct a quasi-cocycle $q\in QZ^1_{as}(\Gamma,\lambda_\Gamma^{\oplus\infty})$ such that $q|_\Lambda\equiv 0$ and which is proper with respect to Λ. By essentially the same techniques as above, this, with some work, ought to imply the almost malnormality of Λ.

4. *Central sequences, asymptotic relative commutants, and inner amenability*

In this section we prove the main results of this paper. First we establish a fairly general theorem which classifies all the central sequences in von Neumann algebras arising from groups (or actions of groups) which admit unbounded quasi-cocycles into (relatively) mixing representations.

4.1. *Non-Gamma factors.*

THEOREM 4.1. *Let Γ be a countable discrete group together with a family of subgroups $\mathcal{G}$ such that Γ satisfies condition $\mathsf{NC}(\mathcal{G})$. Let (A,τ) be any finite von Neumann algebra equipped with a faithful, normal trace τ, and let $\Gamma\curvearrowright(A,\tau)$ be any trace preserving action. Also assume that ω is a free ultrafilter on the positive integers $\mathbb{N}$.*

Then for any asymptotically central sequence $(x_n)_n\in M'\cap M^\omega$ there exists a finite subset $\mathcal{F}\subseteq\mathcal{G}$ such that $(x_n)_n\in\vee_{\Sigma\in\mathcal{F}}(A\rtimes\Sigma)^\omega\vee M$, the von Neumann subalgebra of M^ω generated by M and $(A\rtimes\Sigma)^\omega$ for $\Sigma\in\mathcal{F}$.

Proof. Let q be an arbitrary (anti-)symmetric array into an orthogonal representation π which is non-amenable, and consider the corresponding deformation V_t as defined in §2.3. We fix the notation $A\rtimes\Gamma=M\subset\tilde{M}=(L^\infty(X^\pi)\bar{\otimes}A)\rtimes\Gamma$. We will prove first that V_t converges to the identity on any element of $M'\cap M^\omega$.

So, let $(x_n)_n\in M'\cap M^\omega$ and fix $\varepsilon>0$. Since the representation π is non-amenable, then by Proposition 2.12 so is the Koopman representation $\sigma^\pi:\Gamma\to\mathcal{U}(L^2(X^\pi)\ominus\mathbb{C}1)$.

This implies that one can find a finite subset $K \subset \Gamma$ and $L > 0$ such that for all $\xi \in L^2(\tilde{M}) \ominus L^2(M)$ we have

$$\sum_{k \in K} \|u_k \xi - \xi u_k\|_2 \geq L\|\xi\|_2. \tag{4.1}$$

By Proposition 2.11 above choose $t_\varepsilon > 0$ such that for all $k \in K$ and all $t_\varepsilon \geq t \geq 0$ we have

$$\sup_n \|V_t(u_k x_n u_k^*) - u_k V_t(x_n) u_k^*\|_2 \leq \frac{\varepsilon L}{\sqrt{2}|K|}. \tag{4.2}$$

Using the transversality property (Proposition 2.10) together with (4.1), (4.2), and $\lim_n \|[u_k, x_n]\|_2 = 0$ for all $k \in K$ we see that for all $t_\varepsilon \geq t \geq 0$ we have

$$\begin{aligned}
&\limsup_n \|V_t(x_n) - x_n\|_2 \\
&\quad \leq \limsup_n \sqrt{2}\|e_M^\perp V_t(x_n)\|_2 \\
&\quad \leq \limsup_n \frac{\sqrt{2}}{L} \sum_{k \in K} \|u_k e_M^\perp V_t(x_n) u_k^* - e_M^\perp V_t(x_n)\|_2 \\
&\quad \leq \limsup_n \frac{\sqrt{2}}{L} \left(\sum_{k \in K} \|e_M^\perp V_t(u_k x_n u_k^* - x_n)\|_2 + \sum_{k \in K} \|V_t(u_k x_n u_k^*) - u_k V_t(x_n) u_k^*\|_2 \right) \\
&\quad \leq \limsup_n \frac{\sqrt{2}}{L} \left(\sum_{k \in K} \|[u_k, x_n]\|_2 + \frac{\varepsilon L}{\sqrt{2}} \right) \\
&\quad = \varepsilon,
\end{aligned} \tag{4.3}$$

which proves the desired claim.

From this point the proof breaks into two cases.

Case 1. Let π be an orthogonal representation which is non-amenable and mixing with respect to $\mathcal{G}$. We further assume that π admits an unbounded quasi-cocycle, and let $q \in QZ^1_{as}(\Gamma, \pi)$ be any such one of some defect $D \geq 0$.

In this case we show that the uniform convergence of V_t will be sufficient to locate the asymptotic central sequences in M. Let $(x_n)_n \in M' \cap M^\omega$, and let $\varepsilon > 0$. From the first part there exists $t_\varepsilon > 0$ such that for all $0 \leq |t| \leq t_\varepsilon$ we have

$$\limsup_n \|x_n - V_t(x_n)\|_2 \leq \varepsilon. \tag{4.4}$$

For every $R \geq 0$ denote $B_R = \{g \in \Gamma : \|q(g)\| \leq R\}$, and by P_R the orthogonal projection from $L^2(M)$ onto $\overline{span}\{au_g : a \in A, g \in B_R\}$. Then (4.4) together with a simple computation show that there exists $C \geq 0$ for which

$$\limsup_n \|x_n - P_C(x_n)\|_2 \leq \varepsilon. \tag{4.5}$$

Using the triangle inequality, for every $y \in \mathcal{U}(M)$ we have that

$$\begin{aligned}
\|yP_C(x_n) - P_C(x_n)y\|_2 &\leq \|y(P_C(x_n) - x_n) - (P_C(x_n) - x_n)y\|_2 + \|[y, x_n]\|_2 \\
&\leq 2\|P_C(x_n) - x_n\|_2 + \|[y, x_n]\|_2
\end{aligned}$$

and when this is further combined with (4.5) we get

$$\limsup_n \|yP_C(x_n) - P_C(x_n)y\|_2 \le 2\varepsilon \quad \text{for all } y \in \mathcal{U}(M). \tag{4.6}$$

By Theorem 3.1, for $\gamma \in (B_{2C+2D})^c = \Gamma \backslash B_{2C+2D}$ there exist finite subsets $F, K \subset \Gamma$, $\mathcal{F} \subset \mathcal{G}$ such that

$$\gamma(B_C \backslash F\mathcal{F}K) \cap (B_C \backslash F\mathcal{F}K)\gamma = \emptyset. \tag{4.7}$$

Next we show that $(x_n)_n \in \vee_{\Sigma\in\mathcal{F}}(A \rtimes \Sigma)^\omega \vee M$. Suppose by contradiction this is not the case. Thus by subtracting from $(x_n)_n$ its conditional expectation onto $\vee_{\Sigma\in\mathcal{F}}(A \rtimes \Sigma)^\omega \vee M$ we can assume that $0 \neq (x_n) \perp \vee_{\Sigma\in\mathcal{F}}(A \rtimes \Sigma)^\omega \vee M$. Since the subsets $K, F \subset \Gamma$, $\mathcal{F} \subset \mathcal{G}$ are finite this further implies that

$$\lim_n \|P_{F\mathcal{F}K}(x_n)\|_2 = 0. \tag{4.8}$$

Picking $y = u_\gamma$ with $\gamma \in \Gamma \backslash B_{2C+2D}$ in (4.6), we obtain that

$$\limsup_n \|u_\gamma P_C(x_n) - P_C(x_n)u_\gamma\|_2 \le 2\varepsilon,$$

and using this in combination with (4.8) we have

$$\begin{aligned}
&\limsup_n \|u_\gamma P_{B_C\backslash F\mathcal{F}K}(x_n) - P_{B_C\backslash F\mathcal{F}K}(x_n)u_\gamma\|_2 \\
&\quad \le \limsup_n \|u_\gamma P_{B_C}(x_n) - P_{B_C}(x_n)u_\gamma\|_2 + 2\limsup_n \|P_{F\mathcal{F}K}(x_n)\|_2 \\
&\quad \le 2\varepsilon.
\end{aligned} \tag{4.9}$$

Altogether, relations (4.9), (4.7) and (4.8) lead to the following inequality:

$$\begin{aligned}
4\varepsilon^2 &\ge \limsup_n \|u_\gamma P_{B_C\backslash F\mathcal{F}K}(x_n) - P_{B_C\backslash F\mathcal{F}K}(x_n)u_\gamma\|_2^2 \\
&= \limsup_n (\|u_\gamma P_{B_C\backslash F\mathcal{F}K}(x_n)\|_2^2 + \|P_{B_C\backslash F\mathcal{F}K}(x_n)u_\gamma\|_2^2) \\
&= 2\limsup_n \|P_{B_C\backslash F\mathcal{F}K}(x_n)\|_2^2 \\
&= 2\limsup_n (\|P_{B_C}(x_n)\|_2^2 - \|P_{F\mathcal{F}K}(x_n)\|_2^2) \\
&= \limsup_n 2\|P_{B_C}(x_n)\|_2^2 \\
&= 2(1-\varepsilon)^2,
\end{aligned}$$

which for ε small enough is a contradiction.

Case 2. Let π be a non-amenable representation, and let q be an array associated to π which is proper with respect to $\mathcal{G}$.

Indeed, suppose it was the case that $(x_n)_n \notin \vee_{\Sigma\in\mathcal{F}}(A \rtimes \Sigma)^\omega \vee M$. Let $\xi_n := e_M^\perp V_t(x_n)$, then by essentially the same argument as in [**5**, Theorem 3.2], it would follow that V_t could not converge uniformly to the identity on $M' \cap M^\omega$, a contradiction. □

A well-known theorem of Connes [**7**, Theorem 2.1] shows that property Gamma is equivalent to the existence of a net $\xi_n \in L^2(M) \ominus \mathbb{C}\hat{1}$ of unit-norm vectors such that $\|x\xi_n - \xi_n x\|_2 \to 0$ for all $x \in M$.

COROLLARY 4.2. *If in the previous theorem we assume in addition that the family $\mathcal{G}$ consists only of non-inner amenable subgroups then $M' \cap M^\omega \subseteq A^\omega \rtimes \Gamma$. Therefore if the action $\Gamma \curvearrowright A$ is also free and strongly ergodic then M does not have property* Gamma *of Murray and von Neumann.*

4.2. *Non-inner amenability.* Similar techniques can be applied to provide a fairly large class of groups which are not inner amenable; in particular, this extends some of the results covered by Theorem 4.1. In order to apply the method described in the previous section we need to use a well-known C^*-algebraic characterization of inner amenability which follows from the alternative characterization of inner amenability as stated in the introduction. Namely, Γ is inner amenable if and only if there exists a sequence of unit vectors $(\xi_n)_n \in \ell^2(\Gamma) \ominus \mathbb{C}e$ such that $\lim_n \|x\xi_n - \xi_n x\|_2 = 0$ for all $x \in C^*_r(\Gamma)$. To properly state our result we need to introduce the following definition.

Definition 4.3. Let Γ be a countable group and let $\Sigma < \Gamma$ be a subgroup. We say that Γ is *i.c.c. over* Σ if for every finite subset $F \subset \Gamma$ there exists $\gamma \in \Gamma$ such that $\gamma(F\Sigma F^{-1})\gamma^{-1} \cap F\Sigma F^{-1} = \{e\}$.

We used this terminology only because it naturally extends the classical i.c.c. notion for groups. The next result is probably folklore (and follows from a similar but more general statement such as [**38**, Lemma 2.4]), but we include a proof for the sake of completeness.

PROPOSITION 4.4. *Any countable group Γ is i.c.c. if and only if Γ is i.c.c. over $\Sigma = \{e\}$.*

Proof. One can easily see that the reverse implication holds, so we need only show the forward implication. This follows immediately once we show that for every finite subset $K \subset \Gamma\backslash\{e\}$ there exists $\gamma \in \Gamma$ such that $\gamma K \gamma^{-1} \cap K = \emptyset$. We proceed by contradiction, so suppose there exists a finite set $K_0 \subset \Gamma\backslash\{e\}$ such that for all $\gamma \in \Gamma$ we have that

$$\gamma K_0 \gamma^{-1} \cap K_0 \neq \emptyset. \tag{4.10}$$

Consider the Hilbert space $\mathcal{H} = \ell^2(\Gamma\backslash\{e\})$ and denote by ξ the characteristic function of K_0. Since K_0 is finite then $\xi \in \mathcal{H}$. From (4.10), a simple calculation shows that

$$\langle u_\gamma \xi u_\gamma^{-1}, \xi\rangle \geq \frac{1}{|K_0|} > 0 \quad \text{for all } \gamma \in \Gamma. \tag{4.11}$$

Therefore if we denote by $\mathcal{K} \subset \mathcal{H}$ the closed, convex hull of the set $\{u_\gamma \xi u_{\gamma^{-1}} : \gamma \in \Gamma\}$ and denote by ζ the unique $\|\cdot\|_2$-minimal element in $\mathcal{K}$, then from (4.11) we have that $\langle \zeta, \xi\rangle \geq 1/|K_0| > 0$, in particular $\zeta \neq 0$. Hence, if we decompose $\zeta = \sum_{\gamma \in \Gamma\backslash\{e\}} \zeta_\gamma \delta_\gamma$, there exists $\lambda \in \Gamma\backslash\{e\}$ such that $\zeta_\lambda \neq 0$. On the other hand, by uniqueness, ζ satisfies that $u_\gamma \zeta u_{\gamma^{-1}} = \zeta$ for all $\gamma \in \Gamma$, whence we have $0 \neq \zeta_\lambda = \zeta_{\gamma\lambda\gamma^{-1}}$ for all $\gamma \in \Gamma$. Since $\zeta \in \ell^2(\Gamma)$ it follows that the orbit under conjugation $\{\gamma\lambda\gamma^{-1} : \gamma \in \Gamma\}$ is finite thus contradicting the i.c.c. assumption on Γ. □

With these notations at hand we are ready to state the main theorem.

THEOREM 4.5. *Let Γ be an i.c.c., countable, discrete group together with a family of subgroups $\mathcal{G}$ so that Γ is i.c.c. over every subgroup $\Sigma \in \mathcal{G}$. If Γ satisfies condition* $\mathsf{NC}(\mathcal{G})$, *then Γ is not inner amenable.*

Since any multiple of the left-regular representation of any non-amenable group is both non-amenable and mixing, the theorem shows that every non-amenable i.c.c. group Γ satisfying $QH^1_{as}(\Gamma, \lambda_\Gamma^{\oplus\infty}) \neq \emptyset$ is not inner amenable; in particular, $L\Gamma$ does not have property *Gamma* of Murray and von Neumann. By [**16**, Theorem 1.4] this covers all non-amenable groups which admit hyperbolically embedded subgroups, so our result recovers Theorem 8.2 (f) from [**8**]. The result also demonstrates that all groups with positive first ℓ^2-Betti number are either finite or non-inner amenable since for any such group Γ it holds that $H^1(\Gamma, \lambda_\Gamma) \neq 0$, cf. [**33**]. Finally, we point out that in the case that Γ has a non-amenable orthogonal representation π which admits a proper symmetric array (i.e. Γ belongs to the class $\mathcal{QH}$), then the fact that Γ is not inner amenable is already contained in [**5**, Proposition 1.7.5].

Proof of Theorem 4.5. We will proceed by contradiction, so suppose Γ is inner amenable. Thus there exists a sequence $(\xi_n)_n \in \ell^2(\Gamma) \ominus \mathbb{C}e$ of unit vectors such that for every $x \in C_r^*(\Gamma)$ we have

$$\lim_{n\to 0} \|x\xi_n - \xi_n x\|_2 = 0. \tag{4.12}$$

Let π be a non-amenable representation, and let q be any (anti-)symmetric array associated to π. As in the previous theorem, let $M = L\Gamma$ and let $\tilde{M} = L^\infty(Y^\pi) \rtimes \Gamma$ be the Gaussian construction associated with π. Consider $V_t : L^2(M) \to L^2(\tilde{M})$ for $t \in \mathbb{R}$, the associated path of unitaries as defined in §2.3. Using the non-amenability of π, the same spectral gap argument as in Theorem 4.1 shows that $\lim_{t\to 0}(\sup_n \|e_M^\perp V_t(\xi_n)\|_2) = 0$. By the transversality property (Proposition 2.10) this gives that $\lim_{t\to 0}(\sup_n \|\xi_n - V_t(\xi_n)\|_2) = 0$. Then a simple calculation shows that for every $\varepsilon > 0$ there exists $C \geq 0$ such that

$$\sup_n \|\xi_n - P_{B'_C}(\xi_n)\|_2 \leq \varepsilon. \tag{4.13}$$

As before, we have denoted by $P_{B'_C}$ the orthogonal projection from $\ell^2(\Gamma)$ onto the Hilbert subspace $\ell^2(B'_C)$ with $B'_C = \{\lambda : \|q(\lambda)\| \leq C, \lambda \neq e\}$ being the ball of radius C centered and pierced at the identity element e. Using the triangle inequality, relations (4.12) and (4.13) show that for every $\gamma \in \Gamma$ we have

$$\limsup_n \|u_\gamma P_{B'_C}(\xi_n) - P_{B'_C}(\xi_n) u_\gamma\|_2 \leq 2\varepsilon. \tag{4.14}$$

If we write $\xi_n = \sum_{\eta\in\Gamma} \xi^n_\eta \delta_\eta$ with $\xi^n_\eta \in \mathbb{C}$ then (4.14) gives the following estimates:

$$\begin{aligned}
4\varepsilon^2 &\geq \limsup_n \Bigg\| \sum_{\eta\in B'_C} \xi^n_\eta \delta_{\gamma\eta} - \xi^n_\eta \delta_{\eta\gamma} \Bigg\|_2^2 \\
&= \limsup_n \Bigg(\Bigg\| \sum_{s\in\gamma B'_C \setminus B'_C\gamma} \xi^n_{\gamma^{-1}s} \delta_s \Bigg\|_2^2 + \Bigg\| \sum_{s\in B'_C\gamma \setminus \gamma B'_C} \xi^n_{s\gamma^{-1}} \delta_s \Bigg\|_2^2 \\
&\quad + \Bigg\| \sum_{s\in\gamma B'_C \cap B'_C\gamma} (\xi^n_{\gamma^{-1}s} - \xi^n_{s\gamma^{-1}}) \delta_s \Bigg\|_2^2 \Bigg)
\end{aligned}$$

$$= \limsup_n \Big(2 \sum_{s \in B'_C} |\xi^n_s|^2 + \sum_{s \in \gamma B'_C \cap B_C \gamma} |\xi^n_{\gamma^{-1}s} - \xi^n_{s\gamma^{-1}}|^2 - \sum_{s \in \gamma B'_C \cap B'_C \gamma} (|\xi^n_{\gamma^{-1}s}|^2 + |\xi^n_{s\gamma^{-1}}|^2) \Big).$$

Since ξ_n are unital vectors then the previous estimate together with (4.13) show that

$$\begin{aligned} 4\varepsilon^2 + \limsup_n \sum_{s \in \gamma B'_C \cap B'_C \gamma} \Big(|\xi^n_{\gamma^{-1}s}|^2 + |\xi^n_{s\gamma^{-1}}|^2 \Big) \\ \geq \limsup_n \Big(2\| P_{B'_C}(\xi_n) \|_2^2 + \sum_{s \in \gamma B'_C \cap B'_C \gamma} |\xi^n_{\gamma^{-1}s} - \xi^n_{s\gamma^{-1}}|^2 \Big) \\ \geq 2(1-\varepsilon^2). \end{aligned}$$

Altogether, the previous inequalities imply that for every $\gamma \in \Gamma$ we have

$$\limsup_n \Big(\sum_{s \in (\gamma B'_C \gamma^{-1} \cup \gamma^{-1} B'_C \gamma) \cap B'_C} |\xi^n_s|^2 \Big) \geq 2(1 - 3\varepsilon^2). \tag{4.15}$$

Since $\sum_s |\xi^n_s|^2 = \|\xi_n\|^2 = 1$ we conclude that, for every $\gamma \in \Gamma$ we have

$$\limsup_n \| P_{A_\gamma}(\xi_n) \|_2^2 = \limsup_n \Big(\sum_{s \in A_\gamma} |\xi^n_s|^2 \Big) \geq 1 - 6\varepsilon^2. \tag{4.16}$$

Here for every $\gamma \in \Gamma$ we have denoted $A_\gamma = \gamma B'_C \gamma^{-1} \cap B'_C$ and for a set $\Omega \subset \Gamma$ we denoted by P_Ω the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Omega)$.

Claim. We have assumed that Γ is i.c.c. over every $\Sigma \in \mathcal{G}$ and admits a map $q : \Gamma \to \mathcal{H}$ such that either: (1) q is an array associated to π which is proper with respect to $\mathcal{G}$; or (2) π is mixing with respect to $\mathcal{G}$ and q is an unbounded quasi-cocycle. In either case we claim that there exist $\gamma \in \Gamma$ together with finite subsets $\mathcal{G}_o \subset \mathcal{G}$ and $F \subset \Gamma$ such that $A_\gamma \subseteq \bigcup_{\Sigma \in \mathcal{G}_o} F \Sigma F$. When q is a proper array this is straightforward because for every $\gamma \in \Gamma$ we have $A_\gamma \subset B_C$ and, by the properness assumption, the latter is contained in a finite union of finitely many left-right translates of groups in $\mathcal{G}$. For the other case, denote by D the defect of the quasi-cocycle q and fix $\gamma \in \Gamma \setminus B_{2C+2D}$. Applying Theorem 3.1 it follows that $A_\gamma \gamma = \gamma B'_C \cap B'_C \gamma$ is contained in a set which is small with respect to $\mathcal{G}$. This further implies that one can find finite sets $\mathcal{G}_o \subset \mathcal{G}$ and $F \subset \Gamma$ such that $A_\gamma \subseteq \bigcup_{\Sigma \in \mathcal{G}_o} F \Sigma F$, as desired.

Using our claim, after passing to a subsequence of ξ_n, the inequality (4.16) implies that for all n we have

$$\| P_{\bigcup_{\Sigma \in \mathcal{G}_o} F \Sigma F}(\xi_n) \|_2^2 = \sum_{s \in \bigcup_{\Sigma \in \mathcal{G}_o} F \Sigma F} |\xi^n_s|^2 \geq 1 - 6\varepsilon^2.$$

Since $\mathcal{G}_o$ is finite, by passing one more time to a subsequence of ξ_n there exists $\Sigma \in \mathcal{G}_o$ such that for all n we have

$$\| P_{F\Sigma F \setminus \{e\}}(\xi_n) \|_2^2 = \| P_{F \Sigma F}(\xi_n) \|_2^2 = \sum_{s \in F \Sigma F} |\xi^n_s|^2 \geq \frac{1 - 6\varepsilon^2}{|\mathcal{G}_o|} := M_\varepsilon > 0, \tag{4.17}$$

where $P_{F\Sigma F}$ denotes the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(F\Sigma F)$. Next we claim that from the assumption that Γ is i.c.c. over every group in $\mathcal{G}$ one can construct inductively two infinite sequences $(F_\ell)_\ell$, $(G_\ell)_\ell$ of finite subsets of Γ such that $(F_\ell\Sigma G_\ell)\backslash\{e\}$ are pairwise disjoint sets and $\limsup_n \|P_{F_\ell\Sigma G_\ell\backslash\{e\}}(\xi_n)\|_2^2 \geq M_\varepsilon$ for all $\ell\in\mathbb{N}$. Using Parseval's identity for every $\ell\in\mathbb{N}$ we have that $1=\limsup_n \|\xi_n\|_2^2 \geq \limsup_n \sum_{i=1}^{\ell}\|P_{F_i\Sigma G_i\backslash\{e\}}(\xi_n)\|_2^2 \geq \ell M_\varepsilon$, which is a contradiction when ℓ is arbitrarily large, whence Γ is not inner amenable.

In the remaining part of the proof of this case we show the claim above by induction on ℓ. Since the case $\ell=1$ follows immediately from (4.17) by letting $F_1=G_1=F$, we only need to show the induction step. So assume that for $1\leq i\leq\ell$ we have constructed finite subsets F_i, $G_i\subset\Gamma$ such that sets $(F_i\Sigma G_i)\backslash\{e\}$ are pairwise disjoint and

$$\limsup_n \|P_{F_i\Sigma G_i\backslash\{e\}}(\xi_n)\|_2^2 \geq M_\varepsilon \quad \text{for all } 1\leq i\leq\ell. \tag{4.18}$$

Now we will indicate how to build the subsets $F_{\ell+1}$, $G_{\ell+1}\subset\Gamma$ with the required properties. Since F_i and G_i are finite sets then so are $F'=\bigcup_{i=1}^{\ell}F_i$ and $G'=\bigcup_{i=1}^{\ell}G_i$ and from the assumption there exists $\mu\in\Gamma$ such that $\mu(F'\Sigma G')\mu^{-1}\cap F'\Sigma G'=\{e\}$.

Using the projection formula $P_{\gamma\Omega\gamma^{-1}}(\xi)=u_\gamma P_\Omega(u_{\gamma^{-1}}\xi u_\gamma)u_{\gamma^{-1}}$ for $\gamma\in\Gamma$, $\xi\in\ell^2(\Gamma)$, and $\Omega\subseteq\Gamma$ in combination with the triangle inequality, $\lim_n\|u_{\mu^{-1}}\xi_n u_\mu-\xi_n\|_2=0$, and (4.18) we see that

$$\begin{aligned}
\limsup_n \|P_{\mu(F'\Sigma G')\mu^{-1}\backslash\{e\}}(\xi_n)\|_2 &= \limsup_n \|P_{\mu(F'\Sigma G')\mu^{-1}}(\xi_n)\|_2\\
&\geq \limsup_n(\|u_\mu P_{F'\Sigma G'}(\xi_n)u_{\mu^{-1}}\|_2\\
&\quad -\|P_{\mu(F'\Sigma G')\mu^{-1}}(\xi_n)-u_\mu P_{F'\Sigma G'}(\xi_n)u_{\mu^{-1}}\|_2)\\
&= \limsup_n(\|P_{F'\Sigma G'}(\xi_n)\|_2-\|P_{F'\Sigma G'}(u_\mu\xi_n u_\mu-\xi_n)\|_2)\\
&\geq \limsup_n(\|P_{F'\Sigma G'}(\xi_n)\|_2-\|u_{\mu^{-1}}\xi_n u_\mu-\xi_n\|_2)\\
&\geq \limsup_n \|P_{F'\Sigma G'}(\xi_n)\|_2-\lim_n\|u_{\mu^{-1}}\xi_n u_\mu-\xi_n\|_2\\
&\geq M_\varepsilon.
\end{aligned}$$

Altogether, this computation and the choice of x show that the sets $F_{\ell+1}=\mu F'$ and $G_{\ell+1}=G'\mu^{-1}$ satisfy the required conditions.

As a corollary we recover and generalize a result of de la Harpe and Skandalis.

PROPOSITION 4.6. (de la Harpe and Skandalis [**14**]) (1) *If Γ is a lattice in a real, connected, semi-simple Lie group G with trivial center and no compact factors, then Γ is not inner amenable.* (2) *In general, let $G=G_1\times G_2$ be a unimodular l.c.s.c. group such that G_1 is topologically i.c.c.† and has property (HH). Then any i.c.c. irreducible lattice $\Gamma<G$ is not inner amenable.*

† That is, for any compact neighborhood $e\in K\subset G$ of the identity, and any neighborhood $U\ni e$, there exists $g_1,\ldots,g_n\in G$ such that $g_1Kg_1^{-1}\cap\cdots\cap g_nKg_n^{-1}\subset U$.

Proof. Note that in case (1) Γ is i.c.c. as a consequence of Borel density, cf. [**13**]. Without loss of generality we may assume Γ is irreducible, as any lattice is a product of such. In the case (1), we may further assume that G does not have property (T); otherwise, this would imply that Γ is an i.c.c. property (T) group, therefore not inner amenable. Hence, G has a factor with property (HH), cf. [**30**]. Since G is without compact factors, it is topologically i.c.c.; thus, we have reduced case (1) to case (2).

So, now assume we are in the general situation of case (2). By Example 2.6 and Theorem 4.5, we need only show that Γ is i.c.c. relative to any subgroup Σ such that the projection into G_1 is pre-compact. This is true by the topological i.c.c. property and the irreducibility which implies that the image of Γ under the projection is dense in G_1. □

4.3. *Around property Gamma.* If M is a separable II_1 factor, a trivial consequence of M not having property Gamma is that M cannot be written as an infinite tensor product of non-scalar finite factors. The next result shows that not having property Gamma implies a stabilized version of this property; to be precise, M not having property Gamma implies that $M\bar{\otimes}R$ is not isomorphic to a infinite tensor product of non-amenable factors. As usual, R denotes the hyperfinite II_1 factor. We remark that under the stronger assumption that each factor in the tensor is non-Gamma, indecomposability follows by [**37**, Theorem 4.1].

THEOREM 4.7. *Let M be a II_1 factor which does not have property Gamma. If $\{N_i : i \in I\}$ is any countable collection of non-amenable II_1 factors such that $M\bar{\otimes}R \cong \bar{\bigotimes}_{i\in I} N_i$ then I is a finite set.*

Proof. Suppose by contradiction that $|I| = \infty$; hence, there exists an infinite sequence $I_n \subset I$ of finite subsets such that $I_n \subset I_{n+1}$ and $\bigcup_n I_n = I$. Denote $J_n = I\backslash I_n$, $N(J_n) = \bar{\bigotimes}_{i\in J_n} N_i$, and $N(I_n) = \bar{\bigotimes}_{i\in I_n} N_i$. Fix ω a free ultrafilter on $\mathbb{N}$. Applying a spectral gap argument, we will show that there exists $s \in \mathbb{N}$ such that $N(J_s)^\omega \subseteq M\bar{\otimes}R^\omega$.

To do this, we note that by [**7**, Theorem 2.1] there exists a finite subset $F \subset \mathcal{U}(M)$ and $C > 0$ such that for all $x \in M\bar{\otimes}R$ we have

$$\sum_{u\in F} \|xu - ux\|_2^2 \geq C\|E_R(x) - x\|_2^2. \tag{4.19}$$

Since M is the inductive limit of $N(I_n)$ as $n \to \infty$ and $N(I_n)$ commute with $N(J_n)$ for all n then using (4.19) together with some basic approximations and the triangle inequality we obtain the following: for every $\varepsilon > 0$ there exist $s_\varepsilon \in \mathbb{N}$ such that for every $x \in (N(J_{s_\varepsilon}))_1$ we have $\|E_R(x) - x\|_2 \leq \varepsilon$. If we let ε to be small enough, by applying Popa's intertwining techniques from [**34**] we obtain that a corner of $N(J_{s_\varepsilon})$ intertwines into R inside M. This however is a contradiction because no corner of a non-amenable factor can be intertwined into an amenable von Neumann algebra. Therefore I cannot be infinite, and we have finished. □

Following [**40**] a factor M is called *asymptotically abelian* if there exists a sequence of automorphisms $\theta_n \subset \mathrm{Aut}(M)$ such that $\lim_n \|\theta_n(x)y - y\theta_n(x)\|_2 = 0$ for all $x, y \in M$. Next we will show that using the previous techniques one can provide a fairly large class of algebras that are not asymptotically abelian. In particular the result provides many new examples of McDuff factors which are not asymptotically abelian, enlarging the class of examples found in [**40**].

PROPOSITION 4.8. *Let M be a II_1 factor which does not have property Gamma. If P is any amenable finite factor, then the factor $M\bar{\otimes}P$ is not asymptotically abelian.*

Proof. Suppose by contradiction that $N := M\bar{\otimes}P$ is asymptotically abelian; thus, there exists a sequence of automorphisms $\theta_k \in \mathrm{Aut}(N)$ such that $\|\theta_k(x)y - y\theta_k(x)\|_2 \to 0$ as $k \to \infty$ for all $x, y \in N$. This means that $(\theta_k(x))_k \in N' \cap N^\omega$ for all $x \in N$. As in the previous proposition we have $(\theta_k(x))_k \in P^\omega$ for all $x \in N$ and all $1 \leq i \leq n$. This implies that of every $(y_k)_k \in M^\omega$ we have that $\lim_k \|\theta_k(x)y_k - y_k\theta_k(x)\|_2 = 0$ for all $x \in N$. Since the automorphisms θ_n are τ-invariant we obtain that $\lim_k \|x\Phi_k(y_k) - \Phi_k(y_k)x\|_2 = 0$ for all $x \in N$, where $\Phi_k = \theta_k^{-1}$. Thus $\Phi_k(y_k)$ is an asymptotically central sequence, so by the same argument as before we have that $(\Phi_k(y_k))_k \in P^\omega$ for all $(y_k)_k \in M^\omega$. Thus for all $(y_k)_k \in M^\omega$ we have

$$\lim_{k\to\omega} \|E_P(\Phi_k(y_k)) - \Phi_k(y_k)\|_2 = 0. \tag{4.20}$$

Next we show that this is will lead to a contradiction. Since P is amenable then for each $k \in \mathbb{N}$ no corner of $\Phi_k(M)$ can be intertwined in the sense of Popa into P inside N, [**34**]. Thus by [**34**, Theorem 2.3] for each $k \in \mathbb{N}$ there exists a unitary $u_k \in \mathcal{U}(M)$ such that $\|E_P(\Phi_k(u_k))\|_2 \leq 1/k$. Since u_k is a unitary then using this in combination with (4.20) we have that

$$\begin{aligned}1 &= \lim_k \|\Phi_k(u_k)\|_2 \leq \lim_k(\|E_P(\Phi_k(u_k))\|_2 + \|\Phi_k(u_k) - E_P(\Phi_k(u_k))\|_2)\\ &\leq \lim_k(1/k + \|\Phi_k(u_k) - E_P(\Phi_k(u_k))\|_2) = 0,\end{aligned}$$

which is a contradiction. □

5. *Further remarks*

5.1. *Thompson's groups F, T, and V.* Around the mid 1960s Richard Thompson considered three groups of piecewise linear bijections of the interval preserving the dyadic rationals which he denoted F, T, and V.

These groups have being intensively studied ever since as they display intricate algebraic and analytic properties which provide critical insight into various problems in the theory of discrete groups. For instance V, T, and the commutator of F are rare examples of infinite, finitely presented, simple groups. It is well known that T and V are non-amenable as they contain a copy of the free group with two generators. A long-standing open problem formulated by Geoghegan in the late 1970s asks whether F is amenable. (We point out that it is not even known whether F is exact.) A negative answer would provide a new (finitely presented) counterexample to the famous von Neumann conjecture stating that a discrete group is amenable if and only if it does not contain a copy of the free group with two generators. This conjecture was famously disproved by Ol'shanskii [**24**] and Ol'shanskii and Sapir in the finitely presented case [**25**].

In connection to this, Jolissaint was able to prove that F is inner amenable (even that $L(F)$ is a McDuff factor), [**17**, **18**].

QUESTION 5.1. *Are the groups V and T inner amenable?*

A possible approach to the question, at least in the case of V, follows from the work of Farley [**10**] who showed that there exists a proper 1-cocycle $c : V \to \ell^2(V/V_0)$ where V_0 is the subgroup of V which acts by the identity on the interval $[0, 1/2)$. If one could show that the group V_0 is *not* co-amenable in V then Theorem 4.5 above would automatically give that V is not inner amenable. However, we are not able to determine whether V_0 is co-amenable in V.

After the first author posed this question at a conference, Uffe Haagerup and Kristian Olesen realized that the methods they had recently developed to show that V and T generate non-Gamma factors could be generalized to show that T and V are not inner amenable, [**11**]. We remark that their techniques are completely different from the method suggested above, relying on a non-trivial combinatorial analysis of the commutators of a particular non-amenable subgroup $\Lambda < T < V$.

5.2. *Central sequences and simplicity of group C^*-algebras.* In [**44**], the author poses the question of whether the reduced C^*-algebra of an i.c.c., countable, discrete group Γ is simple if the group has positive first ℓ^2-Betti number. In light of the fact that, by [**8**, Theorem 8.12], C^*-simplicity is known for groups admitting a non-degenerate hyperbolically embedded subgroup, we propose that this ought to be true in the following more general context.

QUESTION 5.2. *If Γ is an i.c.c., countable, discrete group satisfying condition* NC*, is $C^*_r(\Gamma)$ simple?*

By the work of Akemann and Pedersen [**1**] C^*-simplicity of an i.c.c. group is equivalent to the non-existence of certain central sequences in $C^*_r(\Gamma)$. Let Γ be an i.c.c., countable, discrete group. Recall, a *central sequence* in $C^*_r(\Gamma)$ is a bounded sequence (z_n) such that $\|xz_n - z_nx\|_\infty \to 0$ for all $x \in C^*_r(\Gamma)$. A central sequence is said to be trivial if there exists a sequence of scalars (c_n) such that $\|z_n - c_n 1\|_\infty \to 0$. A central sequence is (z_n) is said to be *summable* if each $z_n \geq 0$ and $\sum_n z_n = z \in L\Gamma$, where the sum is understood to converge in the strong topology. It is important to note though $L(\mathbb{F}_2)$ does not have property Gamma, $C^*_r(\mathbb{F}_2)$ is rife with non-trivial central sequences [**1**, Theorem 2.4]. However, $C^*_r(\mathbb{F}_2)$ has no non-trivial *summable* central sequences as a consequence of [**1**, Theorems 3.1 and 3.3].

It would be highly interesting to investigate whether any C^*-algebraic 'rigidity' techniques can be developed which, similarly to the von Neumann algebraic techniques used above, could be used to rule out the presence of summable (norm) central sequences. Specifically, it would be interesting to know whether 'spectral gap' type phenomena exist at the C^*-level. Take the following concrete situation: Γ is a non-amenable countable, discrete group, $\Gamma \curvearrowright X$ is a Bernoulli action, and $B := L^\infty(X) \rtimes_r \Gamma$ is the reduced crossed product. Suppose that x_n is a positive, summable sequence in B such that $\|ax_n - x_na\| \to 0$ for all $a \in C^*_r(\Gamma)$. Is it the case that there exists a sequence $x'_n \in C^*_r(\Gamma) \subset B$ so that $\|x_n - x'_n\| \to 0$?

Acknowledgements. We would like to thank D. Osin and J. Peterson for useful conversations on the subject of this paper. We also thank A. Ioana for useful comments on the manuscript. I. C. was supported in part by NSF Grant #1263982. T. S. was supported by an NSF RTG Assistant Adjunct Professorship.

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