Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Shortest Paths

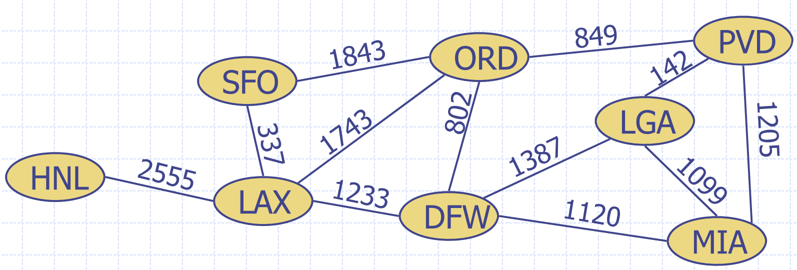
Lightning strike, 2009. U.S. government image. NOAA.

1

1

Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports

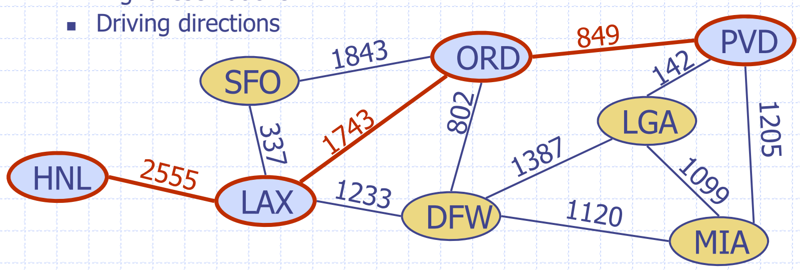


2

2

Shortest Paths

- Given a weighted graph and two vertices ***u*** and ***v***, we want to find a path of minimum total weight between ***u*** and ***v***.
  - Length of a path is the sum of the weights of its edges.
- Example:
  - Shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions



3

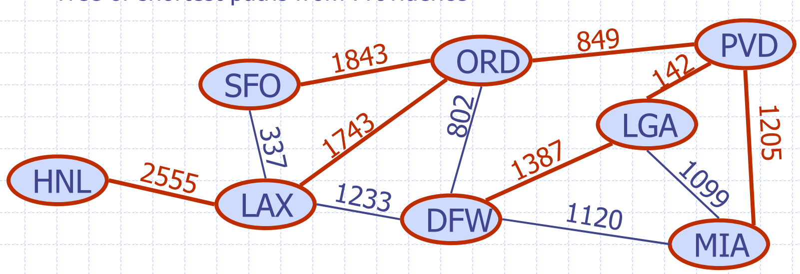
3

Shortest Path Properties

Property 1:
A subpath of a shortest path is itself a shortest path

Property 2:
There is a tree of shortest paths from a start vertex to all the other vertices

Example:
Tree of shortest paths from Providence



4

4

Dijkstra's Algorithm

- The distance of a vertex v from a vertex s is the length of a shortest path between s and v
- Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s
- Assumptions:
  - the graph is connected
  - the edges are undirected
  - the edge weights are nonnegative
- We grow a "cloud" of vertices, beginning with s and eventually covering all the vertices
- We store with each vertex v a label $D[v]$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- At each step
  - We add to the cloud the vertex u outside the cloud with the smallest distance label, $D[u]$
  - We update the labels of the vertices adjacent to u

5

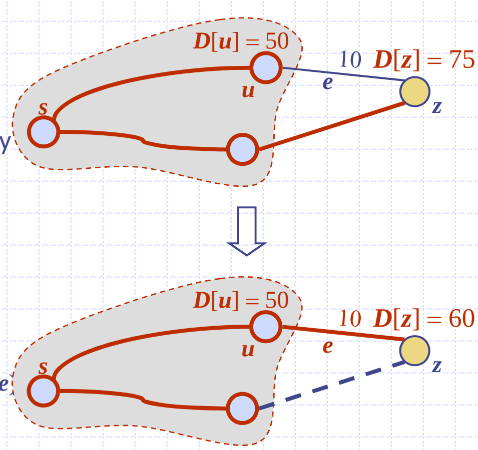
5

Edge Relaxation

- $D[v]$ = the shortest distance of v from s found so far
- Consider an edge $e = (u, z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud
- The relaxation of edge e updates distance $d(z)$ as follows:

$D[z] \leftarrow \min\{D[z], D[u] + weight(e)\}$

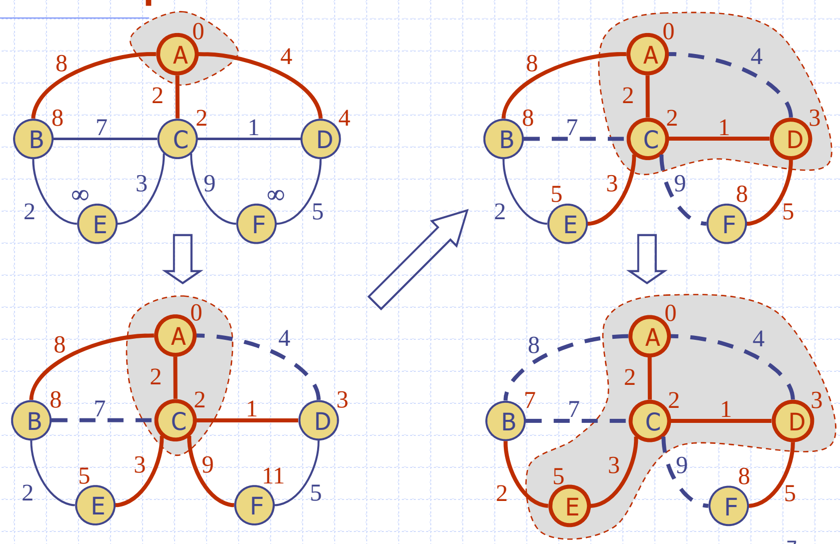
$D[v]$ = the distance of v from s



6

6

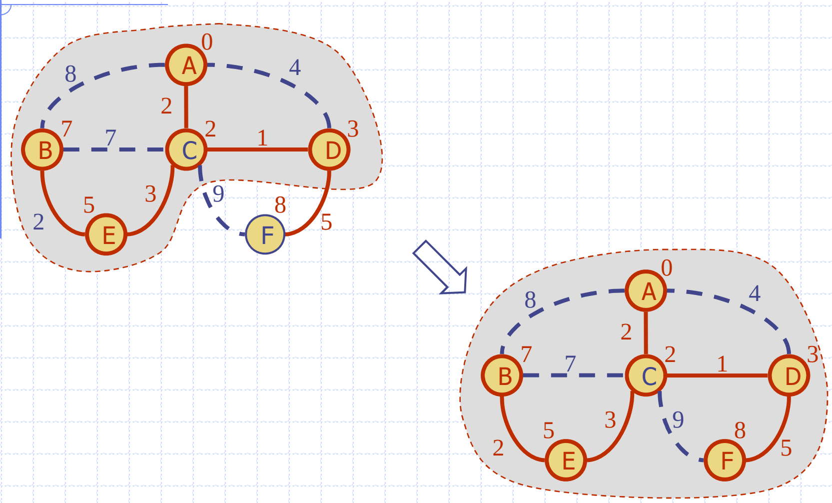
Example



7

7

Example (cont.)



8

8

Dijkstra's Algorithm: Details

Alg. Dijkstra(V, E)

Input: A weighted directed graph $G=(V, E)$, $V=\{1, 2, \dots, n\}$;

Output: The distance from vertex 1 to every other vertex in G;

parent of y

```
X={1}; Y←V−{1}; D[1]←0;
for y←2 to n
    if (y is adjacent to 1) { D[y]←length[1, y]; p[y] ← 1 }
    else D[y]←∞;
for j← 2 to n
    Let y ∈ Y s.t. D[y] is minimum; // y = argmin_{y∈Y} { D[y] }
    X←X ∪ {y};          // add vertex y to cloud X
    Y←Y − {y};          //delete vertex y from Y
    for each edge (y, w) in E // edge relaxation
        if (w ∈ Y and D[y]+length[y, w]<D[w])
            { D[w]←D[y]+length[y, w]; p[w] ← y; }
```

9

Analysis of Dijkstra's Algorithm

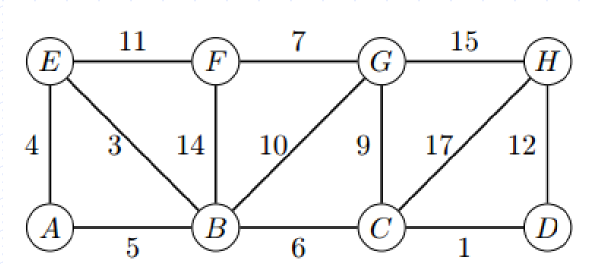
- Graph operations
  - We find all the incident edges once for each vertex
- Label operations
  - We set/get the distance and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Dijkstra's algorithm runs in $O((n+m)\log n)$ time provided the graph is represented by the adjacency list/map structure
  - Recall that $\sum_v \deg(v) = 2m$
- The running time can also be expressed as $O(m \log n)$ since the graph is connected (m > n-2).

10

10

Possible Quiz Question

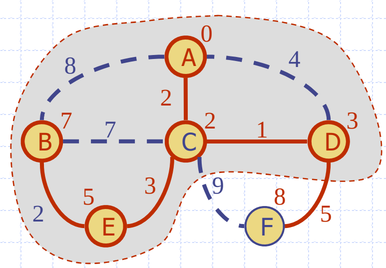
Find the shortest paths from A to all other vertices and draw the tree found by Dijkstra's Algorithm.



11

Why Dijkstra's Algorithm Works

- Dijkstra's algorithm is based on the greedy method. It adds vertices to cloud by increasing distance.
  - Suppose it didn't find all shortest distances. Let w be the first wrong vertex the algorithm processed.
  - When the previous node, u, on the true shortest path was considered, its distance was correct
  - But the edge (u,w) was relaxed at that time!
  - Thus, so long as D[w]≥D[u], w's distance cannot be wrong. That is, there is no wrong vertex



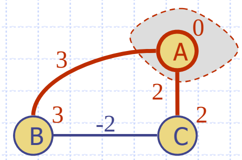
(u,w) = (D,F) in this example

12

12

Why It Doesn't Work for Negative-Weight Edges

- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.
  - If a node with a negative incident edge were to be added late to the cloud, it could mess up distances for vertices already in the cloud.
  - Example: The shortest path from A to C is through B: the distance is 3 + -2 = 1.



13

13

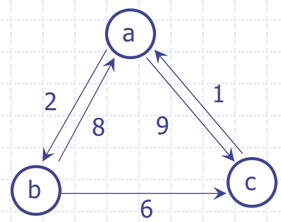
The All-Pairs Shortest Path Problem

- Let $G=(V, E)$ be a directed graph in which each edge (i, j) has a non-negative length w[i, j]. If there is no edge from vertex i to vertex j, then w[i, j]=∞.
- The problem is to find the minimal distance from each vertex to all other vertices, where the distance from vertex x to vertex y is the sum of the edge lengths in a path from x to y.

14

The All-Pairs Shortest Path Problem

- Example:



Weight:

w	a	b	c
a	0	2	9
b	8	0	6
c	1	-	0

Distance:

D	a	b	c
a	0	2	8
b	7	0	6
c	1	3	0

15

Design a Dynamic Programming Solution

- How are the subproblems formulated?
- Where are the solutions stored?
- How are the base values computed?
- How do we compute each entry from other entries in the table?
- What is the order in which we fill in the table?

16

Two DP algorithms for All-pairs shortest paths

- Both are correct. Both produce correct values for all-pairs shortest paths.
- The difference is the subproblem formulation, and hence in the running time.
- Be prepared to provide one or both of these algorithms, and to be able to apply it to an input (on some exam, for example).

17

Dynamic Programming

First attempt: let {1,2,…,n} denote the set of vertices.

Subproblem formulation:
M[i,j,k] = min length of any path from i to j that uses *at most* k edges.

All paths have at most n-1 edges, so $1 \le k \le n-1$.

When k=1, M[i,j,1] = w[i,j], the edge weight from i to j.

Minimum paths from i to j are found in M[i,j,n-1]

- Question: How to set M[i,j,k] from other entries?

18

- How to set M[i,j,k] from other entries, for k>1?
- Consider a *minimum weight* path from i to j that has at most k edges.
  - Case 1: The minimum weight path has at most k-1 edges.
    - M[i,j,k] = M[i,j,k-1]
  - Case 2: The minimum weight path has exactly k edges.
    - M[i,j,k] = min{ M[i,x,k-1] + w(x,j) : x in V}
- Combining the two cases:

  M[i,j,k] = min{min{M[i,x,k-1] + w(x,j) : x in V}, M[i,j,k-1]}

19

Finishing the design

- How are the subproblems defined?

  Subproblem formulation:

  M[i,j,k] = min length of any path from i to j that uses *at most* k edges.
- Where is the answer stored?
  - Minimum paths from i to j are found in M[i,j,n-1]
- How are the base values computed?
  - When k=1, M[i,j,1] = w[i,j], the edge weight from i to j.
- How do we compute each entry from other entries?
  - M[i,j,k] = min{min{M[i,x,k-1] + w(x,j) : x in V}, M[i,j,k-1]}
- What is the order in which we fill in the matrix?
  - For k from 1 to n-1, compute M[i,j,k].
- Running time?

20

Pseudo-Code and Complexity Analysis

```
for j = 1 to n  for i = 1 to n
        M[i,j,1] = w[i,j];
for k = 2 to n-1
   for j = 1 to n
     for i = 1 to n {
        // M[i,j,k] = min{min{M[i,x,k-1] + w(x,j) : x in V}, M[i,j,k-1]}
        minx = M[i,j,k-1];
        for x = 1 to n
            if (minx > M[i,x,k-1] + w(x,j)) minx = M[i,x,k-1] + w(x,j);
        M[i,j,k] = minx;
     }
```

- How many entries do we need to compute? $O(n^3)$
  $1 \le i \le n$; $1 \le j \le n$; $1 \le k \le n-1$
- How much time does it take to compute each entry? $O(n)$
- Total time: $O(n^4)$ Total space: $O(n^3)$ (or $O(n^2)$)

21

Next DP approach: Marshall's Algorithm

- Try a new subproblem formulation!
- Q[i,j,k] = minimum weight of any path from i to j that uses internal vertices drawn from {1,2,…,k}.

22

Designing a DP solution

- How are the subproblems formulated?
  - Q[i,j,k] = minimum weight of any path from i to j that uses internal vertices (other than i and j) drawn from {1,2,…,k}.
- Where is the answer stored?
  - Q[i,j,n] stores the min length from i to j.
- How are the base values computed?
  - Base cases: Q[i,j,0] = w[i,j] for all i,j
- How do we compute each entry from other entries?
- What is the order in which we fill in the matrix?

23

Solving subproblems

- Q[i,j,k] = minimum weight of any path from i to j that uses internal vertices drawn from {1,2,…,k}.
- Such minimum cost path either includes vertex k or does not include vertex k.
- If the minimum cost path P includes vertex k, then you can divide P into the path P_1 from i to k, and P_2 from k to j.
- What is the weight of P_1? Q[i,k,k-1] (why??).
- What is the weight of P_2? Q[k,j,k-1] (why??).
- Thus the weight of P is Q[i,k,k-1] + Q[k,j,k-1].

24

Marshall's Algorithm

```
for j = 1 to n
      for i = 1 to n
         Q[i,j,0] = w[i,j]
for k= 1 to n
    for j = 1 to n
      for i = 1 to n
         Q[i,j,k] = min{Q[i,j,k-1],
                        Q[i,k,k-1] + Q[k,j,k-1]}
```

- Each entry only takes O(1) time to compute
- There are $O(n^3)$ entries
- Hence, $O(n^3)$ time.
- Total space: $O(n^3)$ (or $O(n^2)$)

25

Reusing the space

```
// Use R[i,j] for Q[i,j,0], Q[i,j,1], …, Q[i,j,n].
for j = 1 to n
    for i = 1 to n
         R[i,j] = w[i,j];
for k= 1 to n
    for j = 1 to n
        for i = 1 to n
           R[i,j] = min{R[i,j], R[i,k] + R[k,j]}
```

Claim: For any k, min path of i to j <= R[i,j] <= Q[i,j,k].

26

How to check negative cycles

```
// Use R[i,j] for Q[i,j,0], Q[i,j,1], …, Q[i,j,n].
for j = 1 to n
    for i = 1 to n
        R[i,j] = w[i,j];
for k= 1 to n
    for j = 1 to n
      for i = 1 to n
          R[i,j] = min{R[i,j], R[i,k] + R[k,j]};
for i = 1 to n
    if (R[i,i] < 0) print("There is a negative cycle");
```

27

How to compute transitive closure

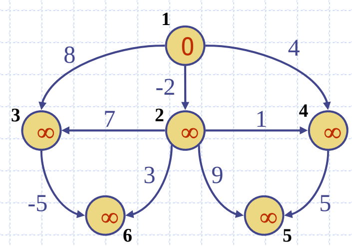
- The relation $R^* = R^1 \cup R^2 \cup R^3 \cup \ldots \cup R^{n-1}$, where n is the number of nodes, is called the transitive closure of R.
- To decide if (a, b) in R*, we need to decide if there is a path from a to b in G = (A, R).

```
// Pre: R[,] is the relation over {1, 2, .., n}
// Post: T is the transitive closure of R.
for j = 1 to n
    for i = 1 to n
        T[i,j] = R[i,j];  // R[.,.] is 0/1 incidence matrix for relation R.
for k= 1 to n
    for j = 1 to n
      for i = 1 to n
          T[i,j] = T[i,j] || (T[i,k] && T[k,j]);
```

28

Shortest Paths in DAG

- We can produce a specialized shortest-path algorithm for directed acyclic graphs (DAGs)
- Works even with negative-weight edges
- Uses topological order
- It doesn’t use any fancy data structures
- It’s much faster than Dijkstra’s algorithm.

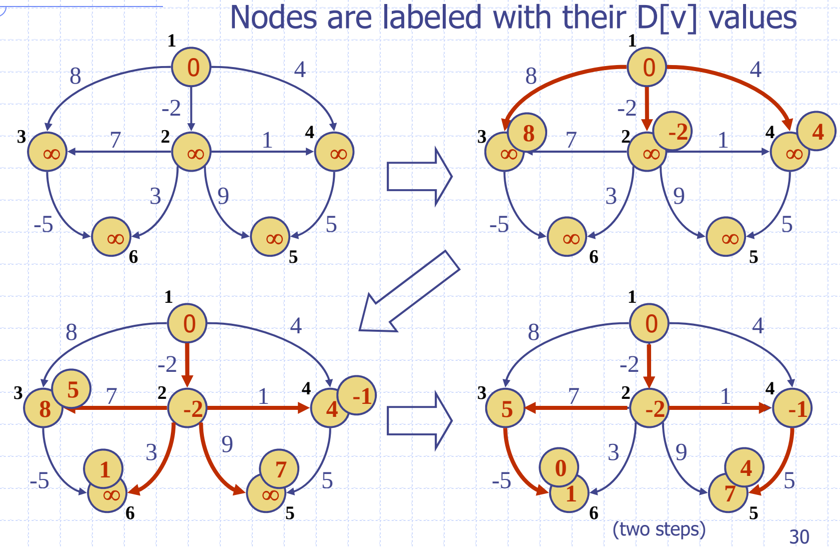


29

29

DAG Example

Nodes are labeled with their D[v] values



30

30

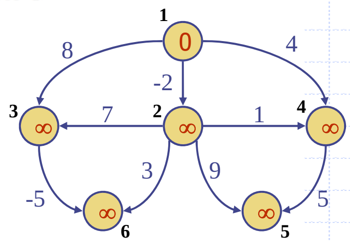
DAG-based Algorithm: Details

Algorithm DAGShortestPaths($\vec{G}, s$):

Input: A weighted directed acyclic graph (DAG) $\vec{G}$ with n vertices and m edges, and a distinguished vertex s in $\vec{G}$

Output: A label $D[u]$, for each vertex u of $\vec{G}$, such that $D[u]$ is the distance from v to u in $\vec{G}$

Compute a topological ordering $(v_1, v_2, \ldots, v_n)$ for $\vec{G}$
$D[s] \leftarrow 0$
for each vertex $u \neq s$ of $\vec{G}$ **do**
    $D[u] \leftarrow +\infty$
for $i \leftarrow 1$ to $n-1$ **do**
    // Relax each outgoing edge from v_i
    for each edge (v_i, u) outgoing from v_i **do**
        if $D[v_i] + w((v_i, u)) < D[u]$ **then**
            $D[u] \leftarrow D[v_i] + w((v_i, u))$
Output the distance labels D as the distances from s.



What is the complexity? Running time: O(n+m). 31

31

Possible Quiz Question

Given a DAG, how to find the longest paths from one vertex to other vertices efficiently?

32