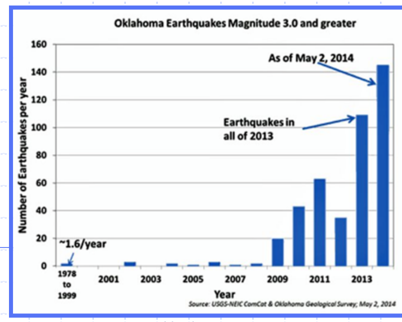
Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Fast Sorting and Selection



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1

1

A Lower Bound for Worst Case

***Theorem*:** Any comparison sort algorithm requires $\Omega(n \lg n)$ comparisons in the worst case.

Proof:

- Suffices to determine the height of a decision tree.
- The number of leaves is at least $n!$ (# outputs)
- The number of internal nodes $\geq n!-1$
- The height is at least $\log(n!-1) = \Omega(n \lg n)$

2

Can we do better?

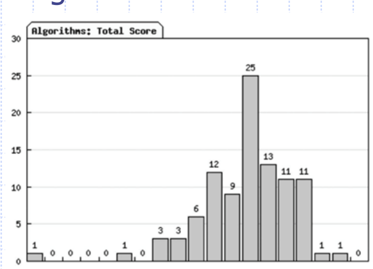
- Linear sorting algorithms
  - Bucket Sort
  - Counting Sort (special case of Bucket Sort)
  - Radix Sort
- Make certain assumptions about the data
- Linear sorts are NOT "comparison sorts"

3

3

Application: Constructing Histograms

- One common computation in data visualization and analysis is computing a **histogram**.
- For example, n students might be assigned integer scores in some range, such as 0 to 100, and are then placed into ranges or "buckets" based on these scores.



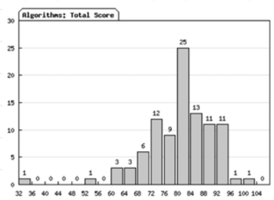
A histogram of scores from a recent Algorithms course.

4

4

Application: An Algorithm for Constructing Histograms

- When we think about the algorithmic issues in constructing a histogram of n scores, it is easy to see that this is a type of sorting problem.
- But it is not the most general kind of sorting problem, since the keys being used to sort are simply integers in a given range.
- So a natural question to ask is whether we can sort these values faster than with a general comparison-based sorting algorithm.
- The answer is "yes." In fact, we can sort them in O(n) time.



5

5

Bucket-Sort

- Let be S be a sequence of n (key, element) items with keys in the range $[0, r-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)

  Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$

  Phase 2: For $i = 0, \ldots, r-1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + r)$ time

  Bucket-sort takes $O(n + r)$ time

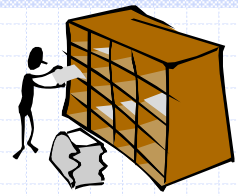
Algorithm bucketSort(S):
Input: Sequence S of entries with integer keys in the range [0, r − 1]
Output: Sequence S sorted in nondecreasing order of the keys
let B be an array of N sequences, each of which is initially empty

for each entry e in S **do**
  k = the key of e
  remove e from S
  insert e at the end of bucket B[k]
for i = 0 to r−1 **do**
  for each entry e in B[i] **do**
    remove e from B[i]
    insert e at the end of S

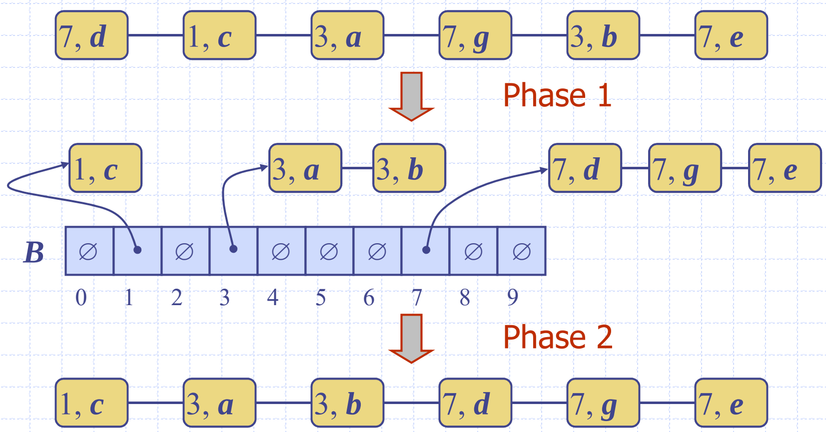
6

6

Example



- Key range [0, 9] (r = 10)



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7

Array-based Implementation: Counting Sort

- Assumptions:
  - n integers which are in the range [0 ... r-1]
  - *r has the same growth rate as n*, that is, r = O(n)
- Idea:
  - For each element x, find the number of occurrences of x and store it in the counter
  - Place several copies of x into its correct position in the output array using the counter as the number of copies.

8

8

Step 1 Find the number of times $A[i]$ appears in A

(i.e., frequencies)

input array A:

3	6	4	1	3	4	1	4

	1	2	3	4	5	6	
allocate C	0	0	0	0	0	0	(r=7)
i=1, A[1]=3	0	0	1	0	0	0	C[A[1]]=C[3]=1
i=2, A[2]=6	0	0	1	0	0	1	C[A[2]]=C[6]=1
i=3, A[3]=4	0	0	1	1	0	1	C[A[3]]=C[4]=1
⋮							
i=8, A[8]=4	2	0	2	3	0	1	C[A[8]]=C[4]=3

C[i] = number of times element i appears in A

```
for (i = 0; i<n; i++)
    C[A[i]]++;
```

9

Step 2

```
1.  int index = 0
2.  For i = 0 to r-1
3.      For j = 1 to C[i]
4.          A[index++] = i
            // Copy value i into the array C[i] times
```

Example:

```
     i:  0 1 2 3 4 5 6
C = [ 0 2 0 2 3 0 1 ]
A = [ 1 1 3 3 4 4 4 6 ]
```

10

10

Properties and Extensions

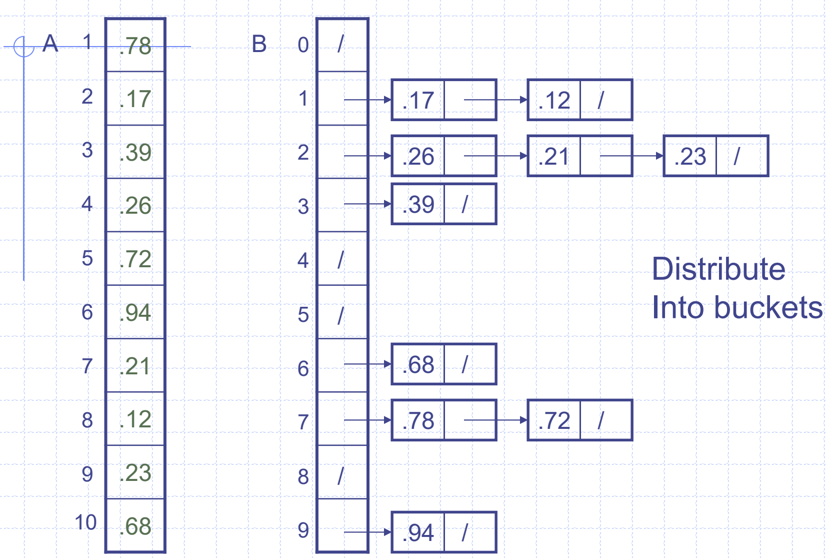
- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- Stable Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions
- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- Float numbers round to integers
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

11

11

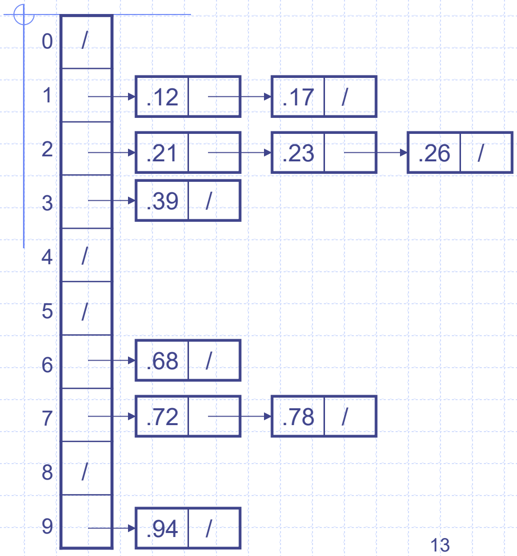
Example - Bucket Sort R = [0..0.99]



12

12

Example - Bucket Sort

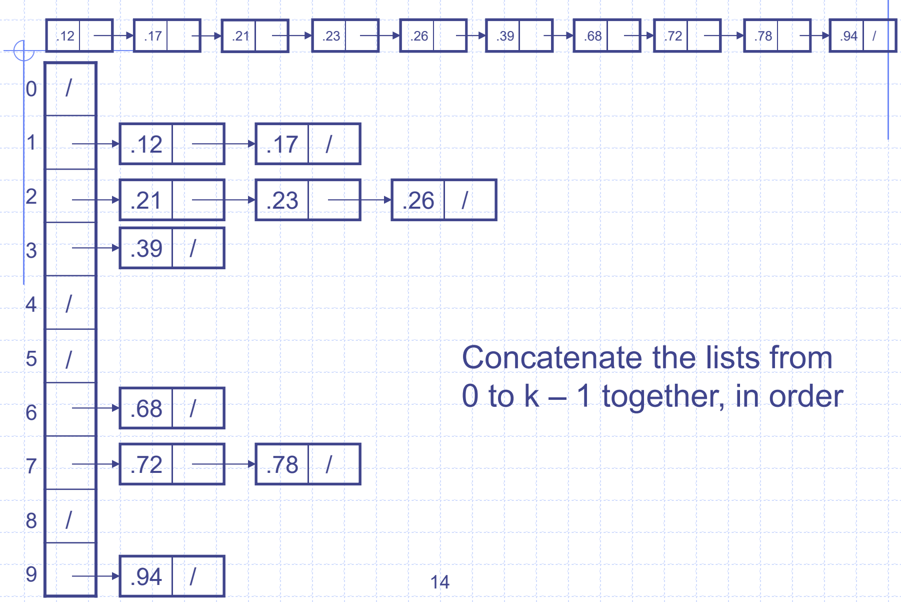


Sort within each bucket: because the mapping from keys to bucket is many-to-one.

13

13

Example - Bucket Sort



Concatenate the lists from 0 to k – 1 together, in order

14

14

Analysis of Extended Bucket Sort

Alg.: BUCKET-SORT(A, n)

for i ← 1 **to** n

do insert A[i] into list B[⌊nA[i]⌋]

O(n)

for i ← 0 **to** r - 1

do sort list B[i] with merge sort

r O(n/r log(n/r))
=O(n log(n/r))
(average case)

concatenate lists B[0], B[1], . . . , B[r -1]
together in order

return the concatenated lists

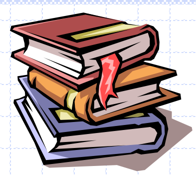
O(n+r)

O(n) (if r=Θ(n))

Note: If the mapping from keys to buckets is 1-to-1, there is no need to sort each bucket, and the time is the worst case, not the average case.

15

Lexicographic Order



- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in 3D space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) <_{\text{lex}} (y_1, y_2, \ldots, y_d)$$
$$\Leftrightarrow$$
$$x_1 < y_1 \vee x_1 = y_1 \wedge (x_2, \ldots, x_d) <_{\text{lex}} (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Example: $(1, 2, 5) <_{\text{lex}} (1, 2, 9)$

16

16

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let *stableSort*(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm *stableSort*, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of *stableSort*

Algorithm *lexicographicSort*(S)
  Input sequence S of d-tuples
  Output sequence S sorted in lexicographic order

  for $i \leftarrow d$ **downto** 1
    stableSort(S, C_i)
    // C_i compares i-th dimension

Example:

(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

17

Correctness of Alg. *lexicographicSort*(S)

Theorem: **Alg.** ***lexicographicSort*(*S*)** sorts S by lexicographic order.

Proof: Induction on d, for d-tuples.

- **Base case**: d=1, stableSort(S, C_1) will do the job.
- **Inductive case:**
  - **Induction hypothesis**: Theorem is true for d' < d.
  - Suppose $(x_1, x_2, \ldots, x_d) <_{lex} (y_1, y_2, \ldots, y_d)$.
  - If $x_1 < y_1$, then the last round places $(x_1, x_2, \ldots, x_d)$ before $(y_1, y_2, \ldots, y_d)$.
  - If $x_1 = y_1$, then $(x_2, \ldots, x_d) <_{lex} (y_2, \ldots, y_d)$.
  - By induction hypothesis, the previous rounds will place $(x_2, \ldots, x_d)$ before $(y_2, \ldots, y_d)$. And we use a stable sort the last round, so $(x_1, x_2, \ldots, x_d)$ goes before $(y_1, y_2, \ldots, y_d)$.

18

18

Radix-Sort

- Radix-sort is a special case of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension.
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, r-1]$
- Radix-sort runs in time $O(d(n+r))$
- If d is constant and r is $O(n)$, then this is $O(n)$.

Algorithm *radixSort*(S, N)
  Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_d, \ldots, x_1)$ and $(x_d, \ldots, x_1) \le (r-1, \ldots, r-1)$ for each tuple $(x_d, \ldots, x_1)$ in S
  Output sequence S sorted in lexicographic order

boolean **procedure** $C_i(x, y)$
  Input $x = (x_d, \ldots, x_1)$,
    $y = (y_d, \ldots, y_1)$, $1 \le i \le d$.
  return $(x_i < y_i)$

for $i \leftarrow 1$ **to** d
  bucketSort(S, r, C_i)

19

19

Radix Sort Example

- Represents keys as d-digit numbers in some base-r
  key = $x_d \ldots x_2x_1$ where $0 \le x_i \le r-1$
- Example: key=479654321
  key = $x_d \ldots x_2x_1$, d=9, r=10 where $0 \le x_i \le 9$

So we can sort US population by SSN in linear time.
How to implement boolean **procedure** $C_i(x, y)$?

20

Radix Sort Example

- Sorting looks at one column at a time
  - For a d digit number, sort the <u>least significant</u> digit first
  - Continue sorting on the <u>next least significant</u> digit, until all digits have been sorted
  - Requires only d passes through the list

326
453
608
835
751
435
704
690

21

21

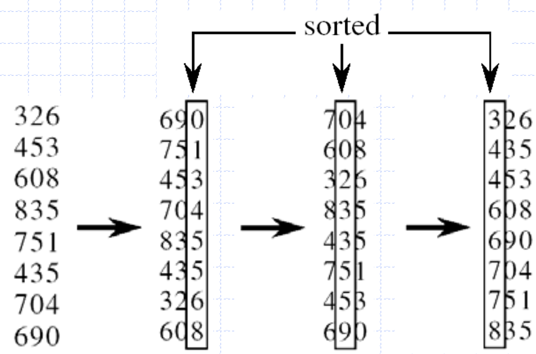
RADIX-SORT

<u>Alg.: RADIX-SORT(*A*, d)</u>

for i ← 1 **to** d

do use a stable bucket sort of array *A* on digit *i*

(stable sort: preserves order of identical elements)



22

22

Analysis of Radix Sort

- Given n numbers of d digits each, where each digit may take up to k possible values, RADIX-SORT correctly sorts the numbers in $O(d(n+r))$
  - One pass of sorting per digit takes $O(n+r)$ assuming that we use **bucket sort**
  - There are d passes (one for each digit)

23

23

Summary: Beating the lower bound

- We can beat the lower bound if we don't base our sort on comparisons:
  - Counting sort for keys in [0..r], r=O(n)
  - Bucket sort for keys which can map to small range of integers (uniformly distributed)
  - Radix sort for keys with a fixed number of "digits"

24

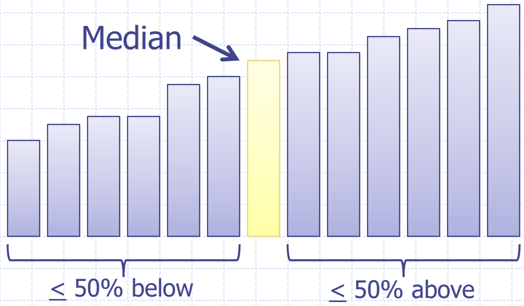
Possible Quiz Question

- Suppose the binary representation of a number $x < 10^9$ is $B_x = \langle b_{30}, b_{29}, \ldots, b_2, b_1 \rangle$ where $0 \le b_i \le 1$. We divide B_x into 6 equal parts, $\langle p_6, p_5, p_4, p_3, p_2, p_1 \rangle$, each part p_i contains 5 bits, representing a number $0 \le p_i \le 31$. Please write an efficient algorithm (in pseudo code) using the radix sort with d=6 and r=32 to sort n numbers which are in the range of 0 to $10^9 - 1$. You may assume that ***bucketSort***(***S***, ***r***, ***C***) is available, where ***S*** is a list of n numbers, ***r*** is the number of buckets, and ***C*** is the comparator for numbers in ***S***. Please analyze the complexity of your algorithm, assuming ***bucketSort***(***S***, ***r***, ***C***) takes O(n+***r***).

25

Finding Medians

- A common data analysis tool is to compute a **median**, that is, a value taken from among n values such that there are at most n/2 values larger than this one and at most n/2 elements smaller.
- Of course, such a number can be found easily if we were to sort the scores, but it would be ideal if we could find medians in O(n) time without having to perform a sorting operation.



26

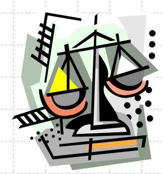
26

Selection: Finding the Median and the *k*th Smallest Element

- The median of a sequence of n sorted numbers $A[1\ldots n]$ is the "middle" element.
- If n is odd, then the middle element is the $(n+1)/2^{th}$ element in the sequence.
- If n is even, then there are two middle elements occurring at positions $n/2$ and $n/2+1$. In this case, we will choose the $n/2^{th}$ smallest element.
- Thus, in both cases, the median is the $\lceil n/2 \rceil^{th}$ smallest element.
- The kth smallest element is a general case.

27

The Selection Problem



- Given an integer k and n elements $x_1, x_2, \ldots, x_n$, taken from a total order, find the k-th smallest element in this set.
- Of course, we can sort the set in O(n log n) time and then index the k-th element.

k=3 7 4 9 <u>6</u> 2 → 2 4 <u>6</u> 7 9

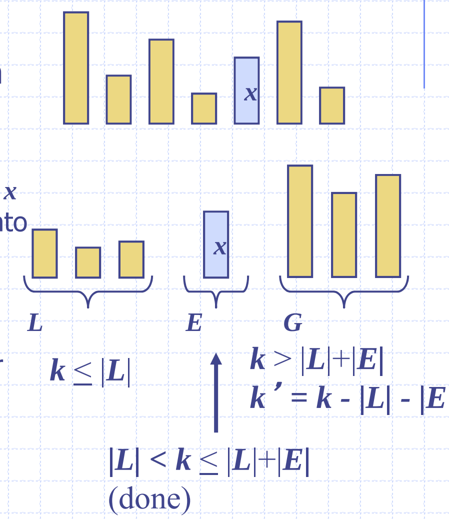
- We want to solve the selection problem faster.

28

28

Quick-Select

- Quick-select is a randomized selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L: elements less than x
    - E: elements equal x
    - G: elements greater than x
  - Search: depending on k, either answer is in E, or we need to recur in either L or G



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29

Pseudo-code

Algorithm quickSelect(S, k):

Input: Sequence S of n comparable elements, and an integer $k \in [1, n]$
Output: The kth smallest element of S

if $n = 1$ **then**
  return the (first) element of S
pick a random element x of S
remove all the elements from S and put them into three sequences:

- L, storing the elements in S less than x
- E, storing the elements in S equal to x
- G, storing the elements in S greater than x.

if $k \le |L|$ **then**
  quickSelect(L, k)
else if $k \le |L| + |E|$ **then**
  return x // each element in E is equal to x
else
  quickSelect$(G, k - |L| - |E|)$

- Note that partitioning takes $O(n)$ time.

30

30

Quick-Select Visualization

- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence

k=5, S=(7 4 9 3 2 6 5 1 8)

k=2, S=(7 4 9 6 5 8)

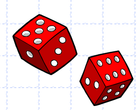
k=2, S=(7 4 6 5)

k=1, S=(7 6 5)

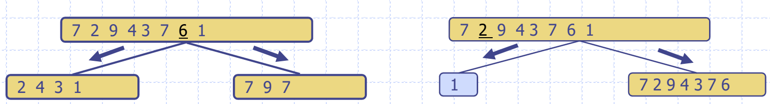
5

31

Expected Running Time



- Consider a recursive call of quick-select on a sequence of size s
  - Good call: the sizes of L and G are each less than $3s/4$
  - Bad call: one of L and G has size greater than $3s/4$



Good call **Bad call**

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:

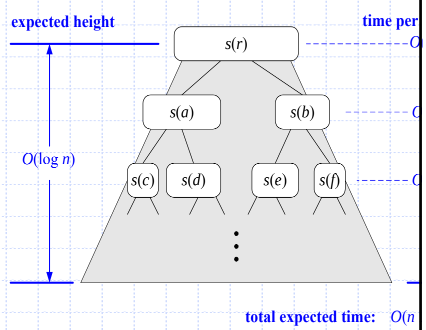
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Bad pivots **Good pivots** **Bad pivots**

32

Expected Running Time, Part 2

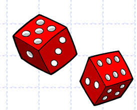
- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of depth i, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O((3/4)^{i/2}n)$
- Thus, the expected running time of quick-sort is $O(n)$



33

33

Expected Running Time



- Let T(n) denote the expected running time of quick-select.
- By Fact #2,
  - $T(n) \le T(3n/4) + bn^*(\text{expected \# of calls before a good call})$
- By Fact #1,
  - $T(n) \le T(3n/4) + 2bn$
- That is, $T(n)$ is a geometric series:
  - $T(n) \le 2bn + 2b(3/4)n + 2b(3/4)^2n + 2b(3/4)^3n + \ldots$
- So $T(n)$ is O(n).
- We can solve the selection problem in O(n) expected time.

34

34

Linear Time Selection Algorithm

- Also called Median Finding Algorithm.
- Find k^{th} smallest element in O(n) time in worst case.
- Uses Divide and Conquer strategy.
- Uses elimination in order to cut down the running time substantially.

35

Deterministic Selection

- If we select an element *m* among *A*, then *A* can be divided in to 3 parts:

  L = { a | a is in A, a < m }
  E = { a | a is in A, a = m }
  G = { a | a is in A, a > m }

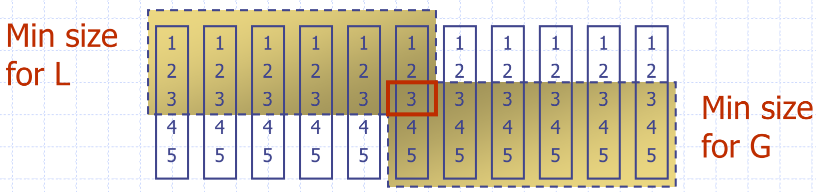
- According to the number elements in *L*, *E*, *G*, there are following three cases. In each case, where is the *k*-th smallest element?

Case 1: $\|L\| >= k$	The k-th element is in L
Case 2: $\|L\| + \|E\| >= k > \|L\|$	The k-th element is in E
Case 3: $\|L\| + \|E\| < k$	The k-th element is in G

36

Deterministic Selection

- We can do selection in O(n) worst-case time.
- Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  - Divide S into n/5 groups of 5 each
  - Find a median in each group
  - Recursively find the median of the "baby" medians.



37

37

Steps to solve the problem

- Step 1: If n is small, for example n<45, just sort and return the k^{th} smallest number in constant time i.e; O(1) time.
- Step 2: Group the given numbers in subsets of 5 in O(n) time.
- Step 3: Sort each of the group in O(n) time. Find median of each group.

38

Example:

- Given a set (……..2,6,8,19,24,54,5,87,9,10,44,32,21,13,3,4, 18,26,36,30,25,39,47,56,71,91,61,44,28………) having n elements.

39

Arrange the numbers in groups of five

…………………	2	54	44	4	25	…………………
…………………	6	5	32	28	39	…………………
…………………	8	87	21	36	47	…………………
…………………	19	9	13	16	56	…………………
…………………	24	10	3	30	71	…………………

40

Sort each group of 5 from top to bottom

2	5	3	4	25
6	9	13	16	39
8	10	21	28	47
19	54	32	30	56
24	87	44	36	71

Each group of 5 is sorted

41

Step 4: Find median of n/5 group medians recursively

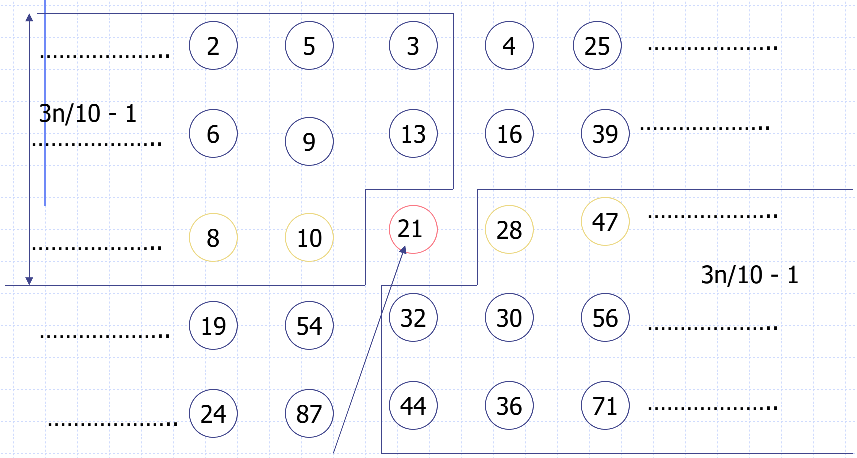
2	5	3	4	25
6	9	13	16	39
8	10	**21**	28	47
19	54	32	30	56
24	87	44	36	71

Median of each group

42

There are $s = n/5$ groups, there are $s/2$ groups on the left of m and $s/2$ groups on the right of m.

So there are $3/2s - 1 = 3n/10 - 1$ numbers less than m and $3n/10 - 1$ numbers greater than m.

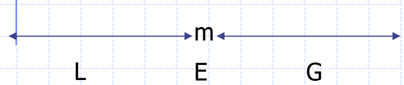


Find m, the median of medians

43

Step 5: Find the sets L, E, and G

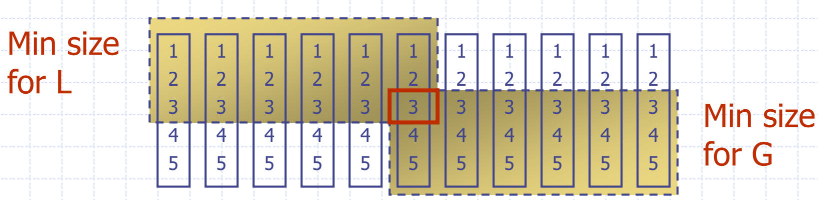
- Compare each (n-1) elements in the top-right and bottom-left regions with the median m and find three sets L, E, and G such that every element in L is smaller than m, every element in E is equal to m, and every element in G is greater than m.



$3n/10 - |E| \leq |L| \leq 7n/10 - |E|$

(|L| is the size or cardinality of L)

$3n/10 - |E| \leq |G| \leq 7n/10 - |E|$



44

Pseudo code: Finding the *k*-th Smallest Element

- **Input**: An array $A[1\ldots n]$ of n elements and an integer k, $1 \le k \le n$;
- **Output**: The kth smallest element in A;
- 1. *select*(A, n, k);

45

Pseudo code: Finding the *k*-th Smallest Element

- *select*(A, n, k)
- 2. if $n < 45$ then sort A and return ($A[k]$);
- 3. Let $q = \lceil n/5 \rceil$. Divide A into q groups of 5 elements each.
-     If 5 does not divide n, then add max element;
- 4. Sort each of the q groups individually and extract its median.
-     Let the set of medians be M.
- 5. $m \leftarrow$ *select*(M, q, $\lceil q/2 \rceil$);
- 6. Partition A into three arrays:
-     $L = \{a \mid a < m\}$, $E = \{a \mid a = m\}$, $G = \{a \mid a > m\}$;
- 7. case
-     $|L| \ge k$: return *select* (L, $|L|$, k);
-     $|L|+|E| \ge k$: return m;
-     $|L|+|E| < k$: return *select*(G, $|G|$, $k-|L|-|E|$);
- 8. end case;

46

Complexity: Finding the *k*-th Smallest Element (Bound time: T(n))

- *select*(A, n, k)
- 2. if $n < 45$ then sort A and return ($A[k]$); — O(1)
- 3. Let $q = \lceil n/5 \rceil$. Divide A into q groups of 5 elements each. — O(n)
-   If 5 does not divide n, then add max element;
- 4. Sort each of the q groups individually and extract its median. — O(n)
-   Let the set of medians be M.
- 5. $m \leftarrow select(M, q, \lceil q/2 \rceil)$; — T(n/5)
- 6. Partition A into three arrays:
-   $L = \{a \mid a < m\}$, $E = \{a \mid a = m\}$, $G = \{a \mid a > m\}$; — O(n)
- 7. case
-   $|L| \geq k$: return *select* (L, $|L|$, k); — T(7n/10)
-   $|L|+|E| \geq k$: return m; — O(1)
-   $|L|+|E| < k$: return *select*(G, $|G|$, $k-|L|-|E|$); — T(7n/10)
- 8. end case;

Summary: T(n) = T(n/5) + T(7n/10) + a*n

47

Analysis: Finding the *k*-th Smallest Element

- What is the best case time complexity of this algorithm?
- O(n) when $|L| < k \leq |L| + |E|$
- T(n): the worst case time complexity of select(A, n, k)

  $$T(n) = T(n/5) + T(7n/10) + a*n$$

- The *k*-th smallest element in a set of *n* elements drawn from a linearly ordered set can be found in $\Theta(n)$ time.

48

Recursive formula

$T(n)= T(n/5) +T(7n/10) + a*n$

We will solve this equation in order to get the complexity.

We guess that $T(n) \le Cn$ for a constant, and then by induction on n.

The base case when $n < 45$ is trivial.

$T(n) = T(n/5) + T(7n/10) + a*n$

$\le C*n/5+ C*7*n/10 + a*n$ (by induction hypothesis)

$= ((2C + 7C)/10 + a)n$

$= (9C/10 + a)n$

$\le Cn$ if $C \ge 9C/10 + a$, or $C/10 \ge a$, or $C \ge 10a$

So we let $C = 10a$.

Then $T(n) \le Cn$.

So $T(n) = O(n)$.

49

Why group of 5??

- If we divide elements into groups of 3 then we will have
  $T(n) = a*n + T(n/3) + T(2n/3)$
  so $T(n)$ cannot be $O(n)$.....

- If we divide elements into groups of more than 5, finding the median of each group will be more, so grouping elements in to 5 is the optimal situation.

50