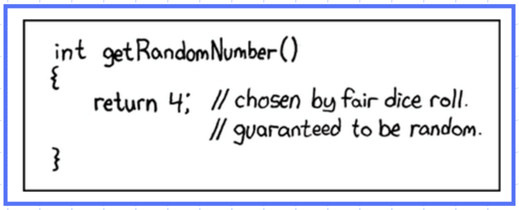
Presentation for use with the textbook Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Hash Tables



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The Search Problem

- Find items with keys matching a given search key
  - *Given an array A, containing n keys, and a search key x, find the index i such as x=A[i]*
  - *As in the case of sorting, a key could be part of a large record.*

example of a record

Key	**other data**

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Special Case: Dictionaries

- **Dictionary** = data structure that supports mainly two basic operations: insert a new item and return an item with a given key.
  - Queries: return information about the set S with key k:
    - get (S, k)
  - Modifying operations: change the set
    - put (S, k): insert new or update the item of key k.
    - remove (S, k) – not very often

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Direct Addressing

- Assumptions:
  - Key values are distinct
  - Each key is drawn from a universe U = {0, 1, . . . , N - 1}
- Idea:
  - Store the items in an array, indexed by keys

- **Direct-address table** representation:
  - An array T[0 . . . N - 1]
  - Each **slot**, or position, in T corresponds to a key in U
  - For an element x with key k, a pointer to x (or x itself) will be placed in location T[k]
  - If there are no elements with key k in the set, T[k] is empty, represented by NIL

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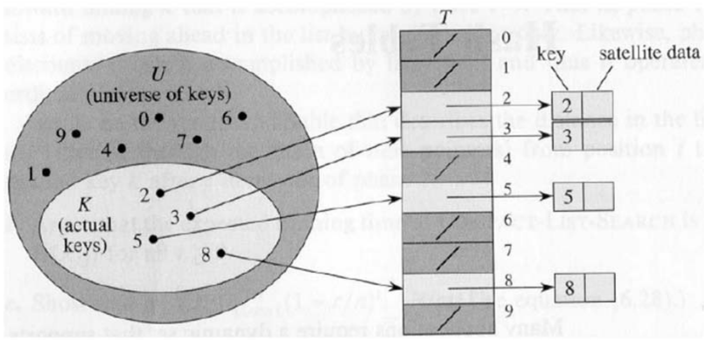
Direct Addressing (cont'd)



(insert/delete in O(1) time)

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Comparing Different Implementations

- Implementing dictionaries using:
  - Direct addressing
  - Ordered/unordered arrays
  - Ordered linked lists
  - Balanced search trees

	put	get
direct addressing	O(1)	O(1)
ordered array	O(N)	O(lgN)
unordered array	O(1)	O(N)
ordered list	O(N)	O(N)
balance search tree	O(lgN)	O(lgN)

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Hash Tables

- When n is much smaller than $max(U)$, where U is the set of all keys, a **hash table** requires much less space than a **direct-address table**
  - Can reduce storage requirements to $O(n)$
  - Can still get $O(1)$ search time, but on the average case, not the worst case

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Hash Tables

  - Use a function h to compute the slot for each key
  - Store the element in slot h(k)
- A **hash function** h transforms a key into an index in a hash table T[0…N-1]:

  $h : U \rightarrow \{0, 1, \ldots, N - 1\}$
- We say that k **hashes** to h(k), hash value of k.
- Advantages:
  - Reduce the range of array indices handled: N instead of $max(U)$
  - Storage is also reduced

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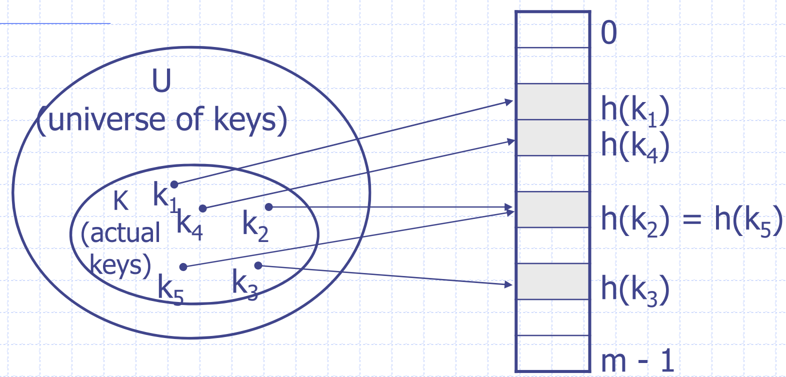
Example: HASH TABLES



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Example

Suppose that the keys are nine-digit social security numbers

Possible hash function

$h(ssn) = sss \bmod 100$ (last 2 digits of ssn)

e.g., if $ssn = 10123411$ then $h(10123411) = 11$)

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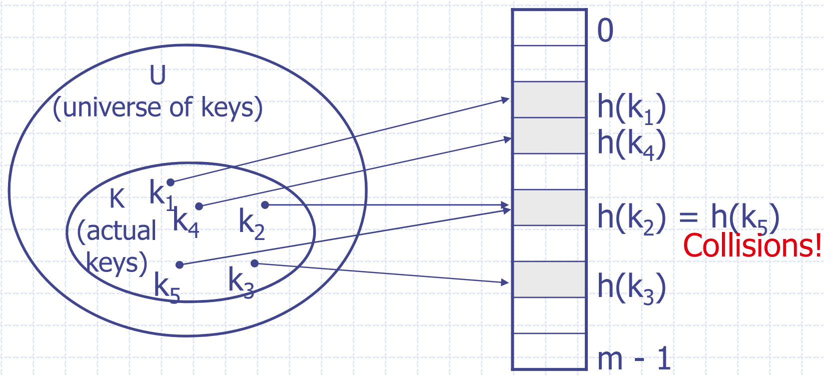
Do you see any problems with this approach?



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Collisions

- Two or more keys hash to the same slot!!
- For a given set of n keys
  - If $n \le N$, collisions may or may not happen, depending on the hash function
  - If $n > N$, collisions will definitely happen (i.e., there must be at least two keys that have the same hash value)
- Avoiding collisions completely is hard, even with a good hash function

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Hash Functions

- A hash function transforms a key into a table address
- **What makes a good hash function?**
  (1) Easy to compute
  (2) Approximates a random function: for every input, every output is equally likely (simple uniform hashing)
- In practice, it is very hard to satisfy the simple uniform hashing property
  - i.e., we don't know in advance the probability distribution that keys are drawn from

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Good Approaches for Hash Functions

- Minimize the chance that closely related keys hash to the same slot
  - Strings such as `stop`, `tops`, and `pots` should hash to different slots
- **Derive a hash value that is independent from any patterns that may exist in the distribution of the keys.**

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The Division Method

- **Idea:**
  - Map a key k into one of the N slots by taking the remainder of k divided by N

    $h(k) = k \bmod N$
- **Advantage**:
  - fast, requires only one operation
- **Disadvantage**:
  - Certain values of N are bad, e.g.,
    - power of 2
    - non-prime numbers

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Example - The Division Method

- If $N = 2^p$, then h(k) is just the least significant p bits of k
  - $p = 1 \Rightarrow N = 2$

    $\Rightarrow h(k) = \{0, 1\}$, least significant 1 bit of k
  - $p = 2 \Rightarrow N = 4$

    $\Rightarrow h(k) = \{0, 1, 2, 3\}$, least significant 2 bits of k
- Choose N to be a prime, not close to a power of 2
  - Column 2: k mod 97
  - Column 3: k mod 100

	N 97	N 100
16838	57	38
5758	35	58
10113	25	13
17515	55	15
31051	11	51
5627	1	27
23010	21	10
7419	47	19
16212	13	12
4086	12	86
2749	33	49
12767	60	67
9084	63	84
12060	32	60
32225	21	25
17543	83	43
25089	63	89
21183	37	83
25137	14	37
25566	55	66
26966	0	66
4978	31	78
20495	28	95
10311	29	11
11367	18	67

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The Multiplication Method

Idea:

- Multiply key k by a constant *A*, where $0 < A < 1$
- Extract the fractional part of k*A*
- Multiply the fractional part by N
- Take the floor of the result

$$h(k) = \lfloor N (kA - \lfloor kA \rfloor) \rfloor$$

- **Disadvantage:** A little slower than division method
- **Advantage:** Value of N is not critical, e.g., typically 2^p

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Hash Functions

- A hash function is usually specified as the composition of two functions:

  Hash code:

  h_1: keys $\rightarrow$ integers

  Compression function:

  h_2: integers $\rightarrow [0, N-1]$

Typically, h_2 is mod N.

- The hash code is applied first, and the compression function is applied next on the result, i.e.,

  $$h(x) = h_2(h_1(x))$$

- The goal of the hash function is to "disperse" the keys in an apparently random way

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Typical Function for H_1

- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)
    $$a_0\, a_1\, \ldots\, a_{n-1}$$
  - We evaluate the polynomial
    $$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$
    $$\ldots + a_{n-1} z^{n-1}$$
    at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice $z = 33$ gives at most 6 collisions on a set of 50,000 English words)
- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horner's rule:
  - The following polynomials are successively computed, each from the previous one in $O(1)$ time
    $$p_0(z) = a_{n-1}$$
    $$p_i(z) = a_{n-i-1} + z p_{i-1}(z)$$
    $$(i = 1, 2, \ldots, n-1)$$
- We have $p(z) = p_{n-1}(z)$
- Good values for z: 33, 37, 39, and 41.

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Compression Functions

- Division:
  - $h_2(y) = y \bmod N$
  - The size N of the hash table is usually chosen to be a prime
  - The reason has to do with number theory and is beyond the scope of this course
- Random linear hash function:
  - $h_2(y) = (ay + b) \bmod N$
  - a and b are random nonnegative integers such that
    $$a \bmod N \neq 0$$
  - Otherwise, every integer would map to the same value b

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Handling Collisions

- We will review the following methods:
  - Separate Chaining
  - Open addressing
    - Linear probing
    - Quadratic probing
    - Double hashing

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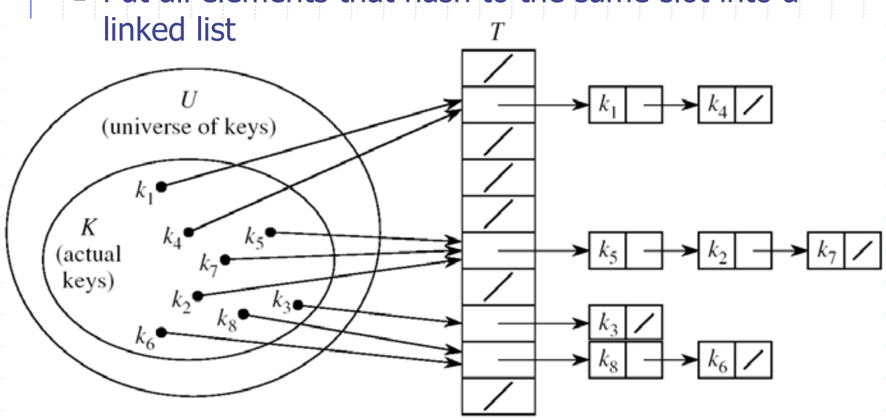
Handling Collisions Using Chaining

- **Idea**:
  - Put all elements that hash to the same slot into a linked list



  - Slot j contains a pointer to the head of the list of all elements that hash to j

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Collision with Chaining

- Choosing the size of the table
  - Small enough not to waste space
  - Large enough such that lists remain short
  - Typically 1/5 or 1/10 of the total number of elements
- How should we keep the lists: ordered or not?
  - Not ordered!
    - Insert is fast
    - Can easily remove the most recently inserted elements

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Insert in Hash Tables

```
Algorithm put(k, v):    // k is a new key
    t = A[h(k)].put(k,v)
    n = n + 1
    return t
```

- Worst-case running time is $O(1)$
- Assumes that the element being inserted isn't already in the list
- It would take an additional search to check if it was already inserted

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Deletion in Hash Tables

```
Algorithm remove(k):
    t = A[h(k)].remove(k)
    if t ≠ null then                {k was found}
        n = n - 1
    return t
```

- Need to find the element to be deleted.
- Worst-case running time:
  - Deletion depends on searching the corresponding list

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Searching in Hash Tables

```
Algorithm get(k):
    return A[h(k)].get(k)
```

- Running time is proportional to the length of the list of elements in slot h(k)

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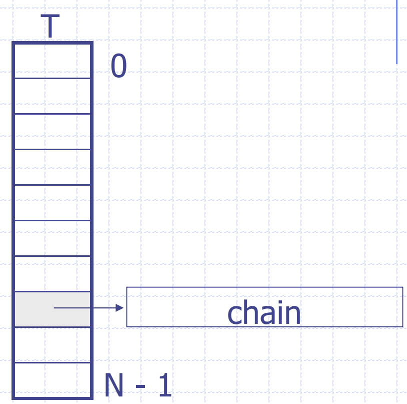
Analysis of Hashing with Chaining: Worst Case

- How long does it take to search for an element with a given key?
- Worst case:
  - All n keys hash to the same slot
  - Worst-case time to search is $\Theta(n)$, plus time to compute the hash function



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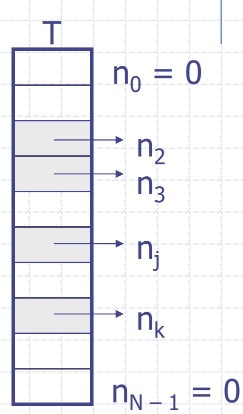
Analysis of Hashing with Chaining: Average Case

- Average case
  - depends on how well the hash function distributes the n keys among the N slots
- **Simple uniform hashing** assumption:
  - Any given element is equally likely to hash into any of the N slots (i.e., probability of collision Pr(h(x)=h(y)), is 1/N)
- Length of a list:

  $T[j].size = n_j, \quad j = 0, 1, \ldots, N - 1$
- Number of keys in the table:

  $n = n_0 + n_1 + \cdots + n_{N-1}$
- Load factor: Average value of n_j:

  $E[n_j] = \alpha = n/N$



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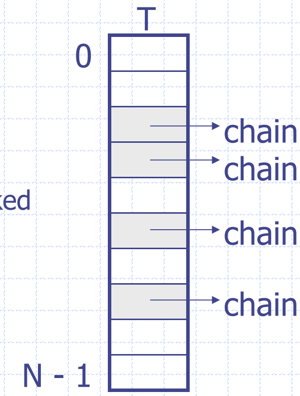
Load Factor of a Hash Table

- Load factor of a hash table T:

  $\alpha = n/N$

  - n = # of elements stored in the table
  - N = # of slots in the table = # of linked lists
- α encodes the average number of elements stored in a chain
- α can be <, =, > 1



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Case 1: Unsuccessful Search (i.e., item not stored in the table)

Theorem An unsuccessful search in a hash table takes expected time $\Theta(1+\alpha)$ under the assumption of simple uniform hashing (i.e., probability of collision Pr(h(x)=h(y)), is 1/N)

Proof

- Searching unsuccessfully for any key k
  - need to search to the end of the list T[h(k)]
- Expected length of the list: $E[n_{h(k)}] = \alpha = n/N$
- Expected number of elements examined in this case is α
- Total time required is:
  - O(1) (for computing the hash function) + $\alpha \rightarrow \Theta(1+\alpha)$

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Case 2: Successful Search

Successful search: $\Theta(1 + \frac{a}{2}) = \Theta(1 + a)$ time on the average

(search half of a list of length a plus $O(1)$ time to compute $h(k)$)

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Analysis of Search in Hash Tables

- If N (# of slots) is proportional to n (# of elements in the table):
- $n = \Theta(N)$
- $\alpha = n/N = \Theta(N)/N = O(1)$

$\Rightarrow$ Searching takes constant time on average

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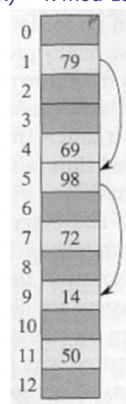
Open Addressing

- If we have enough contiguous memory to store all the keys ⇒ store the keys in the table itself
- No need to use linked lists anymore
- Basic idea:
  - put: if a slot is full, try another one, until you find an empty one
  - get: follow the same sequence of probes
  - remove: more difficult ... (we'll see why)
- Search time depends on the length of the probe sequence!

e.g., insert 14
h(k) = k mod 13



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Generalize hash function notation:

- A hash function contains two arguments now: (i) Key value, and (ii) Probe number

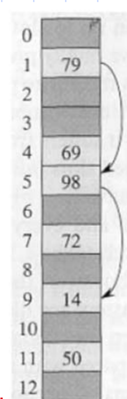
  h(k,p), p=0,1,...,N-1

- Probe sequences

  <h(k,0), h(k,1), ..., h(k,N-1)>

  - Must be a permutation of <0,1,...,N-1>
  - There are N! possible permutations
  - Good hash functions should be able to produce all N! probe sequences

insert 14



Example

<1, 5, 9>

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Common Open Addressing Methods

- Linear probing
- Quadratic probing
- Double hashing

- Note: None of these methods can generate more than N^2 different probing sequences!

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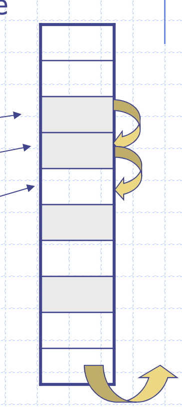
Linear probing

- Idea: when there is a collision, check the next available position in the table (i.e., probing)

  $h(k,i) = (h_1(k) + a*i) \bmod N$

  $i=0,1,2,...$

- First slot probed: $h_1(k)$
- Second slot probed: $h_1(k) + 1$ ($a = 1$)
- Third slot probed: $h_1(k)+2$, and so on



- Can generate N probe sequences maximum, why?

  probe sequence: < h1(k), h1(k)+1 , h1(k)+2 , ...>

wrap around

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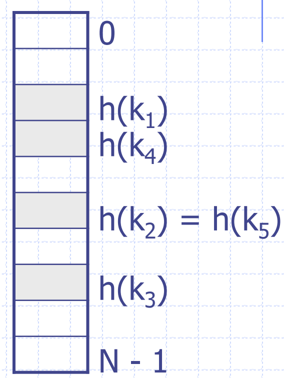
Linear probing: Searching for a key

- Three cases:
  - (1) Position in table is occupied with an element of equal key
  - (2) Position in table is empty
  - (3) Position in table occupied with a different element
- Case 3: probe the next index until the element is found or an empty position is found
- The process wraps around to the beginning of the table



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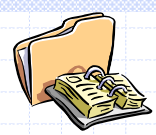

Search with Linear Probing

- Consider a hash table A that uses linear probing
- get(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

Algorithm *get*(k)
  $i \leftarrow h(k)$
  $p \leftarrow 0$
  repeat
    $c \leftarrow A[i]$
    if $c = \varnothing$
      return *null*
    else if $c.getKey() = k$
      return $c.getValue()$
    else
      $i \leftarrow (i + 1) \bmod N$
      $p \leftarrow p + 1$
  until $p = N$
  return *null*

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Quadratic Probing

Less likely to encounter Primary Clustering

$$h(k,i) = (h_1(k) + i^2) \bmod N$$

- Probe sequence:
  - 0^{th} probe = $h(k) \bmod N$
  - 1^{th} probe = $(h(k) + 1) \bmod N$
  - 2^{th} probe = $(h(k) + 4) \bmod N$
  - 3^{th} probe = $(h(k) + 9) \bmod N$
  - . . .
  - i^{th} probe = $(h(k) + i^2) \bmod N$

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Quadratic Probing Example

insert(76) 76%7 = 6 | insert(40) 40%7 = 5 | insert(48) 48%7 = 6 | insert(5) 5%7 = 5 | insert(55) 55%7 = 6

0	
1	
2	
3	
4	
5	
6	76

But… insert(47) 47%7 = 5

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Quadratic Probing: Success guarantee for $\alpha < ½$

- If N is prime and $\alpha < ½$, then quadratic probing will find an empty slot in N/2 probes or fewer, because each probe checks a different slot.
  - Show for all $0 \le i,j \le N/2$ and $i \ne j$
    $(h(x) + i^2) \bmod N \ne (h(x) + j^2) \bmod N$
  - By contradiction: suppose that for some $i \ne j$:
    $(h(x) + i^2) \bmod N = (h(x) + j^2) \bmod N$
    $\Rightarrow i^2 \bmod N = j^2 \bmod N$
    $\Rightarrow (i^2 - j^2) \bmod N = 0$
    $\Rightarrow [(i + j)(i - j)] \bmod N = 0$

Because N is prime `(i-j)` or `(i+j)` must be zero, and neither can be, a contradiction.

Conclusion: For *any* $\alpha < ½$, quadratic probing will find an empty slot; for bigger α, quadratic probing *may* find a slot

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Double Hashing

(1) Use one hash function to determine the first slot

(2) Use a second hash function to determine the increment for the probe sequence

$h(k,i) = (h_1(k) + i\, h_2(k)) \bmod N, \quad i=0,1,\ldots$

- Initial probe: $h_1(k)$
- Second probe is offset by $h_2(k) \bmod N$, so on ...
- Advantage: avoids clustering
- Disadvantage: harder to delete an element
- Can generate N^2 probe sequences maximum

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Double Hashing: Example

$h_1(k) = k \bmod 13$

$h_2(k) = 1 + (k \bmod 11)$

$h(k, i) = (h_1(k) + i\, h_2(k)) \bmod 13$

- Insert key 14:

$h_1(14, 0) = 14 \bmod 13 = 1$

$h(14, 1) = (h_1(14) + h_2(14)) \bmod 13$
$= (1 + 4) \bmod 13 = 5$

$h(14, 2) = (h_1(14) + 2\, h_2(14)) \bmod 13$
$= (1 + 8) \bmod 13 = 9$

0	
1	79
2	
3	
4	69
5	98
6	
7	72
8	
9	14
10	
11	50
12	

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Analysis of Open Addressing

- Ignore the problem of clustering and assume that all probe sequences are equally likely

Unsuccessful retrieval:

Prob(*probe hits an occupied cell*)= a (load factor)

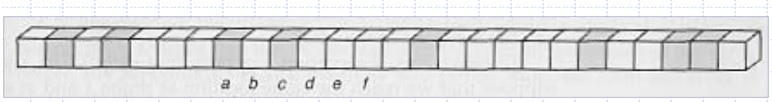
Prob(*probe hits an empty cell*)= $1 - a$

probability that a probe terminates in 2 steps: $a(1-a)$

probability that a probe terminates in k steps: $a^{k-1}(1-a)$

What is the average number of steps in a probe ?

$$E(\#steps) = \sum_{k=1}^{m} k a^{k-1}(1-a) \le \sum_{k=0}^{\infty} k a^{k-1}(1-a) = (1-a)\frac{1}{(1-a)^2} = \frac{1}{1-a}$$



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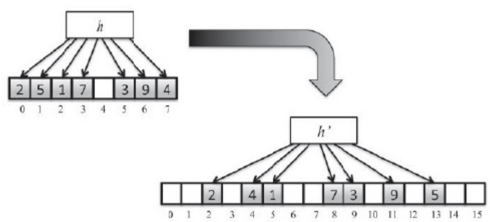
Rehashing

Idea: When the table gets too full, create a bigger table (usually 2x as large) and hash all the items from the original table into the new table.

- When to rehash?
  - half full (α = 0.5)
  - when an insertion fails
  - some other threshold
- Cost of rehashing?



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