Ch02 Data Structures

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Stacks

- Java provides an inbuilt object type called **Stack**.
  
  ```java
  public Stack()
  ```

- Insertions and deletions follow the last-in first-out scheme (LIFO)

- Think of a spring-loaded plate dispenser

- Main stack operations:
  - `push(e)`: inserts an element, e
  - `pop()`: removes and returns the last inserted element

- Auxiliary stack operations:
  - `top()`: returns the last inserted element without removing it
  - `size()`: returns the number of elements stored
  - `isEmpty()`: indicates whether no elements are stored
Example

<table>
<thead>
<tr>
<th>Method</th>
<th>Return Value</th>
<th>Stack Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>push(5)</td>
<td>-</td>
<td>(5)</td>
</tr>
<tr>
<td>push(3)</td>
<td>-</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>size()</td>
<td>2</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>pop()</td>
<td>3</td>
<td>(5)</td>
</tr>
<tr>
<td>isEmpty()</td>
<td>false</td>
<td>(5)</td>
</tr>
<tr>
<td>pop()</td>
<td>5</td>
<td>()</td>
</tr>
<tr>
<td>isEmpty()</td>
<td>true</td>
<td>()</td>
</tr>
<tr>
<td>pop()</td>
<td>null</td>
<td>()</td>
</tr>
<tr>
<td>push(7)</td>
<td>-</td>
<td>(7)</td>
</tr>
<tr>
<td>push(9)</td>
<td>-</td>
<td>(7, 9)</td>
</tr>
<tr>
<td>top()</td>
<td>9</td>
<td>(7, 9)</td>
</tr>
<tr>
<td>push(4)</td>
<td>-</td>
<td>(7, 9, 4)</td>
</tr>
<tr>
<td>size()</td>
<td>3</td>
<td>(7, 9, 4)</td>
</tr>
<tr>
<td>pop()</td>
<td>4</td>
<td>(7, 9)</td>
</tr>
<tr>
<td>push(6)</td>
<td>-</td>
<td>(7, 9, 6)</td>
</tr>
<tr>
<td>push(8)</td>
<td>-</td>
<td>(7, 9, 6, 8)</td>
</tr>
<tr>
<td>pop()</td>
<td>8</td>
<td>(7, 9, 6)</td>
</tr>
</tbody>
</table>
Applications of Stacks

- **Direct applications**
  - Page-visited history in a Web browser
  - Undo sequence in a text editor
  - Chain of method calls in a language supporting recursion

- **Indirect applications**
  - Auxiliary data structure for algorithms
  - Component of other data structures
Method Stacks

- The runtime environment for such a language keeps track of the chain of active methods with a stack.
- When a method is called, the system pushes on the stack a frame containing:
  - Local variables and return value
  - Program counter, keeping track of the statement being executed
- When a method ends, its frame is popped from the stack and control is passed to the method on top of the stack.
- Used for recursion implementation.

```java
class StackExample {
    public static void main(int i) {
        int j = 5;
        foo(i);
    }

    public static void foo(int j) {
        int k;
        k = j+1;
        bar(k);
    }

    public static void bar(int m) {
        ...  // Elaborate implementation
    }
}
```
Array-based Stack

- A simple way of implementing the Stack ADT uses an array
- We add elements from left to right
- A variable keeps track of the index of the top element

**Algorithm size()**

```
return t + 1
```

**Algorithm pop()**

```
if isEmpty() then
    return null
else
    t ← t - 1
    return S[t + 1]
```
Array-based Stack (cont.)

- The array storing the stack elements may become full
- A push operation will then either grow the array or causes the array to double its size.

```
Algorithm push(o)
    if t = S.length - 1 then
        // double S.length
        t ← t + 1
        S[t] ← o
```

[Diagram showing an array-based stack with push operation]
Performance

- Performance
  - Let $n$ be the number of elements in the stack
  - The space used is $O(n)$
  - Each operation runs in time $O(1)$

- Qualifications
  - Trying to push a new element into a full stack causes an implementation-specific exception or
  - Pushing an item on a full stack causes the underlying array to double in size, which implies each operation runs in $O(1)$ amortized time.
Computing Spans (not in book)

- Using a stack as an auxiliary data structure in an algorithm
- Given an array $X$, the span $S[i]$ of $X[i]$ is the maximum number of consecutive elements $X[j]$ before and including $X[i]$, such that $X[j] \leq X[i]$
- Spans have applications to financial analysis
  - E.g., stock at 52-week high

<table>
<thead>
<tr>
<th>X</th>
<th>6</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
Quadratic Algorithm

<table>
<thead>
<tr>
<th>Algorithm $spans1(X, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong> array $X$ of $n$ integers</td>
</tr>
<tr>
<td><strong>Output</strong> array $S$ of spans of $X$</td>
</tr>
<tr>
<td>$S \gets$ new array of $n$ integers</td>
</tr>
<tr>
<td>for $i \gets 0$ to $n - 1$ do</td>
</tr>
<tr>
<td>$s \gets i - 1$</td>
</tr>
<tr>
<td>while $s \geq 0 \land X[s] \leq X[i]$</td>
</tr>
<tr>
<td>$s \gets s - 1$</td>
</tr>
<tr>
<td>$S[i] \gets i - s$</td>
</tr>
<tr>
<td>return $S$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$1 + 2 + \ldots + (n - 1)$</td>
</tr>
<tr>
<td>$1 + 2 + \ldots + (n - 1)$</td>
</tr>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$1$</td>
</tr>
</tbody>
</table>

Algorithm $spans1$ runs in $O(n^2)$ time
Computing Spans with a Stack

- We keep in a stack the indices of the elements larger than the current.
- We scan the array from left to right
  - Let \( i \) be the current index
  - While stack is not empty, we pop indices from the stack until we find index \( j \) such that \( X[j] > X[i] \)
  - If stack is empty, we set \( S[i] \leftarrow i + 1 \) otherwise
  - We set \( S[i] \leftarrow i - j \)
  - We push \( i \) onto the stack

![Array X with stack and S values](image)
Computing Spans with a Stack

- We keep in a stack the indices of the elements larger than the current.
- We scan the array from left to right
  - Let $i$ be the current index
  - While stack is not empty, we pop indices from the stack until we find index $j$ such that $X[j] > X[i]$
  - If stack is empty, we set $S[i] \leftarrow i + 1$ otherwise
  - we set $S[i] \leftarrow i - j$
  - We push $i$ onto the stack

**Algorithm spans2($X, n$)**

$S \leftarrow$ new array of $n$ integers
$A \leftarrow$ new empty stack

for $i \leftarrow 0$ to $n - 1$ do
  while ($\neg A$.isEmpty() $\land$ $X[A$.top$()] \leq X[i]$ )
    $A$.pop()
    if $A$.isEmpty() then
      $S[i] \leftarrow i + 1$
    else
      $S[i] \leftarrow i - A$.top()
  $A$.push($i$)

return $S$
Linear Time Algorithm

- Each index of the array
  - is pushed into the stack exactly one
  - is popped from the stack at most once
- The body of the while-loop is executed at most \( n \) times
- Algorithm \( \text{spans2} \) runs in \( O(n) \) time
- The body of For-loop has \( O(1) \) amortized cost.

```plaintext
Algorithm \( \text{spans2}(X, n) \)

\[
S \leftarrow \text{new array of } n \text{ integers}
A \leftarrow \text{new empty stack}

\text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do}

    \text{while } (\neg A.\text{isEmpty()} \land X[A.\text{top()}] \leq X[i] )

        A.\text{pop()}

    \text{if } A.\text{isEmpty()} \text{ then}
        S[i] \leftarrow i + 1
    \text{else}
        S[i] \leftarrow i - A.\text{top()}

A.\text{push}(i)

\text{return } S
```

\( \Theta \) Each index of the array
\( \Theta \) is pushed into the stack exactly one
\( \Theta \) is popped from the stack at most once
\( \Theta \) The body of the while-loop is executed at most \( n \) times
\( \Theta \) Algorithm \( \text{spans2} \) runs in \( O(n) \) time
\( \Theta \) The body of For-loop has \( O(1) \) amortized cost.
Queues

- In a **Queue**, insertions and deletions follow the first-in first-out scheme (FIFO)
- Insertions are at the “rear” or “end” of the queue and removals are at the “front” of the queue
- Main queue operations:
  - **enqueue***(e)***: inserts an element, *e*, at the end of the queue
  - **dequeue()**: removes and returns the element at the front of the queue
- **Java.util.Queue** is an interface:
  
  ```java
  Queue queueA = new LinkedList();
  enqueue = add, dequeue = remove
  ```

- Auxiliary queue operations:
  - **first()**: returns the element at the front without removing it
  - **size()**: returns the number of elements stored
  - **isEmpty()**: indicates whether no elements are stored
- Boundary cases:
  - Attempting the execution of dequeue or first on an empty queue signals an error or returns **null**
### Example

<table>
<thead>
<tr>
<th>Operation</th>
<th>Output</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>enqueue(5)</td>
<td>–</td>
<td>(5)</td>
</tr>
<tr>
<td>enqueue(3)</td>
<td>–</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>dequeue()</td>
<td>5</td>
<td>(3)</td>
</tr>
<tr>
<td>enqueue(7)</td>
<td>–</td>
<td>(3, 7)</td>
</tr>
<tr>
<td>dequeue()</td>
<td>3</td>
<td>(7)</td>
</tr>
<tr>
<td>first()</td>
<td>7</td>
<td>(7)</td>
</tr>
<tr>
<td>dequeue()</td>
<td>7</td>
<td>()</td>
</tr>
<tr>
<td>dequeue()</td>
<td>null</td>
<td>()</td>
</tr>
<tr>
<td>isEmpty()</td>
<td>true</td>
<td>()</td>
</tr>
<tr>
<td>enqueue(9)</td>
<td>–</td>
<td>(9)</td>
</tr>
<tr>
<td>enqueue(7)</td>
<td>–</td>
<td>(9, 7)</td>
</tr>
<tr>
<td>size()</td>
<td>2</td>
<td>(9, 7)</td>
</tr>
<tr>
<td>enqueue(3)</td>
<td>–</td>
<td>(9, 7, 3)</td>
</tr>
<tr>
<td>enqueue(5)</td>
<td>–</td>
<td>(9, 7, 3, 5)</td>
</tr>
<tr>
<td>dequeue()</td>
<td>9</td>
<td>(7, 3, 5)</td>
</tr>
</tbody>
</table>
Application: Buffered Output

- The Internet is designed to route information in discrete packets, which are at most 1500 bytes in length.
- Any time a video stream is transmitted on the Internet, it must be subdivided into packets and these packets must each be individually routed to their destination.
- Because of vagaries and errors, the time it takes for these packets to arrive at their destination can be highly variable.
- Thus, we need a way of “smoothing out” these variations
Application: Buffered Output

- This smoothing is typically achieved by using a buffer, which is a portion of computer memory that is used to temporarily store items, as they are being produced by one computational process and consumed by another.
- In the case of video packets arriving via the Internet, the networking process is producing the packets and the playback process is consuming them.
- This producer-consumer model is enforcing a first-in, first-out (FIFO) protocol for the packets.
- A data structure that supports this functionality is called a queue.
Additional Applications

- Besides buffering video, queues also have the following applications:

  - Direct applications
    - Waiting lists, bureaucracy
    - Access to shared resources (e.g., printer)
    - Multiprogramming
  
  - Indirect applications
    - Auxiliary data structure for algorithms
    - Component of other data structures
Array-based Queue

- Use an array of size $N$ in a circular fashion
- Two variables keep track of the front and size
  - $f$: index of the front element
  - $sz$: number of stored elements
- When the queue has fewer than $N$ elements, array location $r = (f + sz) \mod N$ is the first empty slot past the rear of the queue

![Diagram showing normal and wrapped-around configurations of an array-based queue](image-url)
Queue Operations

- We use the modulo operator (remainder of division)

**Algorithm size()**
```
return sz
```

**Algorithm isEmpty()**
```
return (sz == 0)
```
Queue Operations (cont.)

- Operation enqueue throws an exception if the array is full.
- One could also grow the underlying array by a factor of 2.

Algorithm `enqueue(o)`

```plaintext
if sz = N then
  signal queue full error
else
  r ← (f + sz) mod N
  Q[r] ← o
  sz ← (sz + 1)
```

![Diagram of queue operations](image)
Queue Operations (cont.)

- Note that operation dequeue returns null if the queue is empty.
- One could alternatively signal an error.

Algorithm `dequeue()`

```plaintext
if isEmpty() then
    return null
else
    o ← Q[f]
    f ← (f + 1) mod N
    sz ← (sz - 1)
    return o
```

```
Q: [ ] [ ] [ ] [ ] [ ]
    0 1 2  f  r
```
```
Q: [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
    0 1 2  f  r
```

```
Application: Round Robin Schedulers

- We can implement a round robin scheduler using a queue Q by repeatedly performing the following steps:
  1. \( e = \text{Q.dequeue()} \)
  2. Service element \( e \)
  3. \( \text{Q.enqueue}(e) \)
Index-Based Lists

- An index-based list supports the following operations:

  - `get(r)`: Return the element of $S$ with index $r$; an error condition occurs if $r < 0$ or $r > n - 1$.
  - `set(r, e)`: Replace with $e$ the element at index $r$ and return it; an error condition occurs if $r < 0$ or $r > n - 1$.
  - `add(r, e)`: Insert a new element $e$ into $S$ to have index $r$; an error condition occurs if $r < 0$ or $r > n$.
  - `remove(r)`: Remove from $S$ the element at index $r$; an error condition occurs if $r < 0$ or $r > n - 1$. 
Example

- A sequence of List operations:

<table>
<thead>
<tr>
<th>Method</th>
<th>Return Value</th>
<th>List Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>add(0, A)</td>
<td>–</td>
<td>(A)</td>
</tr>
<tr>
<td>add(0, B)</td>
<td>–</td>
<td>(B, A)</td>
</tr>
<tr>
<td>get(1)</td>
<td>A</td>
<td>(B, A)</td>
</tr>
<tr>
<td>set(2, C)</td>
<td>“error”</td>
<td>(B, A)</td>
</tr>
<tr>
<td>add(2, C)</td>
<td>–</td>
<td>(B, A, C)</td>
</tr>
<tr>
<td>add(4, D)</td>
<td>“error”</td>
<td>(B, A, C)</td>
</tr>
<tr>
<td>remove(1)</td>
<td>A</td>
<td>(B, C)</td>
</tr>
<tr>
<td>add(1, D)</td>
<td>–</td>
<td>(B, D, C)</td>
</tr>
<tr>
<td>add(1, E)</td>
<td>–</td>
<td>(B, E, D, C)</td>
</tr>
<tr>
<td>get(4)</td>
<td>“error”</td>
<td>(B, E, D, C)</td>
</tr>
<tr>
<td>add(4, F)</td>
<td>–</td>
<td>(B, E, D, C, F)</td>
</tr>
<tr>
<td>set(2, G)</td>
<td>D</td>
<td>(B, E, G, C, F)</td>
</tr>
<tr>
<td>get(2)</td>
<td>G</td>
<td>(B, E, G, C, F)</td>
</tr>
</tbody>
</table>
Array-based Lists

- An obvious choice for implementing the list ADT is to use an array, \( A \), where \( A[i] \) stores (a reference to) the element with index \( i \).

- With a representation based on an array \( A \), the get(\( i \)) and set(\( i, e \)) methods are easy to implement by accessing \( A[i] \) (assuming \( i \) is a legitimate index).
Insertion

- In an operation \textit{add}(i, o), we need to make room for the new element by shifting forward the \( n - i \) elements \( A[i], \ldots, A[n - 1] \)
- In the worst case \((i = 0)\), this takes \( O(n) \) time
Element Removal

- In an operation `remove(i)`, we need to fill the hole left by the removed element by shifting backward the \( n - i - 1 \) elements \( A[i + 1], \ldots, A[n - 1] \).
- In the worst case \( i = 0 \), this takes \( O(n) \) time.
Pseudo-code

- Algorithms for insertion and removal:

Algorithm add\( (r, e) \):

\[
\begin{align*}
\text{if } n &= N \text{ then} \\
\text{return } \text{“Array is full.”} \\
\text{if } r &< n \text{ then} \\
\text{for } i &\leftarrow n - 1, n - 2, \ldots, r \text{ do} \\
A[i + 1] &\leftarrow A[i] \quad \text{// make room for the new element} \\
A[r] &\leftarrow e \\
n &\leftarrow n + 1 \\
\end{align*}
\]

Algorithm remove\( (r) \):

\[
\begin{align*}
e &\leftarrow A[r] \quad \text{// } e \text{ is a temporary variable} \\
\text{if } r &< n - 1 \text{ then} \\
\text{for } i &\leftarrow r, r + 1, \ldots, n - 2 \text{ do} \\
A[i] &\leftarrow A[i + 1] \quad \text{// fill in for the removed element} \\
n &\leftarrow n - 1 \\
\text{return } e \\
\end{align*}
\]
Performance

- In an array-based implementation of a dynamic list:
  - The space used by the data structure is $O(n)$
  - Indexing the element at $i$ takes $O(1)$ time
  - *add* and *remove* run in $O(n)$ time in the worst case

- In an *add* operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one.
Linked Lists

- Linked lists store elements at “nodes” or “positions”.
- Access methods:
  
  first(): Returns the position of the first element of L (or null if empty).
  
  last(): Returns the position of the last element of L (or null if empty).
  
  before(p): Returns the position of L immediately before position p (or null if p is the first position).
  
  after(p): Returns the position of L immediately after position p (or null if p is the last position).
  
  isEmpty(): Returns true if list L does not contain any elements.
  
  size(): Returns the number of elements in list L.
Linked Lists

- **Update methods:**
  
  - `insertBefore(p, e)`: Insert a new element $e$ into $S$ before position $p$ in $S$.
  
  - `insertAfter(p, e)`: Insert a new element $e$ into $S$ after position $p$ in $S$.
  
  - `remove(p)`: Remove from $S$ the element at position $p$.

- **Implementation:**
  
  - The most natural way to implement a positional list is with a doubly-linked list.
Insertion

- Insert a new node, q, between p and its successor.
Deletion

- Remove a node, \( p \), from a doubly-linked list.
Pseudo-code

- Algorithms for insertion and deletion in a linked list:

  **Algorithm** insertAfter($p, e$):
  Create a new node $v$
  $v$.element $\leftarrow$ $e$
  $v$.prev $\leftarrow$ $p$  // link $v$ to its predecessor
  $v$.next $\leftarrow$ $p$.next  // link $v$ to its successor
  ($p$.next).prev $\leftarrow$ $v$  // link $p$’s old successor to $v$
  $p$.next $\leftarrow$ $v$  // link $p$ to its new successor, $v$
  **return** $v$  // the position for the element $e$

  **Algorithm** remove($p$):
  $t$ $\leftarrow$ $p$.element  // a temporary variable to hold the return value
  ($p$.prev).next $\leftarrow$ $p$.next  // linking out $p$
  ($p$.next).prev $\leftarrow$ $p$.prev
  $p$.prev $\leftarrow$ **null**  // invalidating the position $p$
  $p$.next $\leftarrow$ **null**
  **return** $t$
Performance

- A linked list can perform all of the access and update operations for a positional list in constant time.

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>first()</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>last()</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>before($p$)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>after($p$)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>insertBefore($p$, e)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>insertAfter($p$, e)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>remove($p$)</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Exercise A-1.4

An evil king has a cellar containing \( n \) bottles of expensive wine, and his guards have just caught a spy trying to poison the king’s wine. Fortunately, the guards caught the spy after he succeeded in poisoning only one bottle. Unfortunately, they don’t know which one. To make matters worse, the poison the spy used was very deadly; just one drop diluted even a billion to one will still kill someone. Even so, the poison works slowly; it takes a full month for the person to die. Design a scheme that allows the evil king to determine exactly which one of his wine bottles was poisoned in just one month’s time while expending at most \( \Theta(\log n) \) of his taste testers.
Exercise A-1.5

Suppose you are given a set of small boxes, numbered 1 to \( n \), identical in every respect except that each of the first \( i \) contain a pearl whereas the remaining \( n-i \) are empty. You also have two magic wands that can each test whether a box is empty or not in a single touch, except that a wand disappears if you test it on an empty box. Show that, without knowing the value of \( i \), you can use the two wands to determine all the boxes containing pearls using at most \( o(n) \) wand touches. Express, as a function of \( n \), the asymptotic number of wand touches needed.
Exercise A-1.6

Repeat the previous problem assuming that you now have $k$ magic wands, with $k > 2$ and $k < \log n$. Express, as a function of $n$ and $k$, the asymptotic number of wand touches needed to identify all the magic boxes containing pearls.
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure.
- A tree consists of nodes with a parent-child relation.
- **Applications:**
  - Organization charts
  - File systems
  - Programming environments

Diagram:
```
Computers”R”Us
  
Sales  Manufacturing  R&D
  |      |        |
  US    International    Laptops  Desktops
  |          |        |
  Europe  Asia  Canada
```
Tree Terminology

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.

Subtree: tree consisting of a node and its descendants
Tree Operations

- **Accessor methods:**
  
  \[\text{root}()\]: Return the root of the tree.

  \[\text{parent}(v)\]: Return the parent of node \(v\); an error occurs if \(v\) is root.

  \[\text{children}(v)\]: Return a set containing the children of node \(v\).

- **Query methods:**
  
  \[\text{isInternal}(v)\]: Test whether node \(v\) is internal.

  \[\text{isExternal}(v)\]: Test whether node \(v\) is external.

  \[\text{isRoot}(v)\]: Test whether node \(v\) is the root.

- **Generic methods:**
  
  \[\text{size}()\]: Return the number of nodes in the tree.

  \[\text{elements}()\]: Return a set containing all the elements stored at nodes of the tree.

  \[\text{positions}()\]: Return a set containing all the nodes of the tree.

  \[\text{swapElements}(v, w)\]: Swap the elements stored at the nodes \(v\) and \(w\).

  \[\text{replaceElement}(v, e)\]: Replace with \(e\) and return the element stored at node \(v\).
Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

Algorithm $\text{preOrder}(v)$

$\text{visit}(v)$

for each child $w$ of $v$

$\text{preorder}(w)$
Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm $\text{postOrder}(v)$

for each child $w$ of $v$

$\text{postOrder}(w)$

$\text{visit}(v)$

Diagram: A tree structure showing directories and files with their sizes.
Binary Trees

- A binary tree is a tree with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees)
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

Applications:
- arithmetic expressions
- decision processes
- searching
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression \((2 \times (a - 1) + (3 \times b))\)
Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision

- Want a fast meal?
- How about coffee?
  - Yes
  - Starbucks
  - Chipotle
- No
  - On expense account?
  - Yes
  - Gracie’s
  - No
  - Café Paragon
Properties of Proper Binary Trees

- Notation
  - $n$: number of nodes
  - $e$: number of external nodes
  - $i$: number of internal nodes
  - $h$: height

- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$
Binary Tree Operations

- A binary tree extends the Tree operations, i.e., it inherits all the methods of Tree.
- Additional methods:
  - position `leftChild(v)`
  - position `rightChild(v)`
  - position `sibling(v)`
- The above methods return `null` when there is no left, right, or sibling of `p`, respectively.
- Update methods may be defined by data structures implementing the binary tree.
Inorder Traversal

- In an inorder traversal a node is visited after its left subtree and before its right subtree.

Algorithm $inOrder(v)$

```plaintext
if $left(v) \neq \text{null}$
    $inOrder(left(v))$
visit($v$)
if $right(v) \neq \text{null}$
    $inOrder(right(v))$
```
Inorder Traversal

- Application: draw a binary tree from x(v) and y(v):
  - x(v) = inorder rank of v
  - y(v) = depth of v

- Example:
  - v: 1, 2, 3, 4, 5, 6, 7, 8, 9
  - x(v): 2, x, a, –, 1, +, 3, x, b
  - y(v): 2, 1, 3, 2, 3, 0, 2, 1, 2

Algorithm?
Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree

Algorithm `printExpression(v)`

```plaintext
if left(v) ≠ null || right(v) ≠ null
  print("(")
if left(v) ≠ null
  inOrder(left(v))
print(v.element())
if right(v) ≠ null
  inOrder(right(v))
if left(v) ≠ null || right(v) ≠ null
  print(")")
```

```
((2 × (a – 1)) + (3 × b))
```
Evaluate Arithmetic Expressions

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

Algorithm `evalExpr(v)`

```python
if isExternal(v):
    return v.element()
else:
    x ← evalExpr(left(v))
    y ← evalExpr(right(v))
    return v.element(x, y)
```

```
2
- 3 2
  × 5 1
× +
```

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Euler Tour Traversal

- Generic traversal of a tree
- Travel each edge exactly twice.

Euler Tour = \( e_1 e_3 e_4 e_7 e_8 e_4 e_1 e_2 e_5 e_5 e_6 e_6 e_2 \)
Linked Structure for Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT
Linked Structure for Binary Trees

- A node is represented by an object storing:
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT
Array-Based Representation of Binary Trees

- Nodes are stored in an array $A$

Node $v$ is stored at $A[pos(v)]$
- $pos(root) = 0$
- if node is the left child of parent(node), $pos(node) = 2 \cdot pos(parent(node)) + 1$
- if node is the right child of parent(node), $pos(node) = 2 \cdot pos(parent(node)) + 2$
Exercise A-1.11

Given an array, $A$, of $n$ positive integers, each of which appears in $A$ exactly twice, except for one integer, $x$, describe an $O(n)$-time method for finding $x$ using only a single variable besides $A$. 
Exercise A-1.12

Given an array, \( A \), of \( n - 2 \) unique integers in the range from 1 to \( n \), describe an \( \mathcal{O}(n) \)-time method for finding the two integers in the range from 1 to \( n \) that are not in \( A \). You may use only \( \mathcal{O}(1) \) space in addition to the space used by \( A \).

\[
\begin{align*}
1 + 2 + \ldots + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6}
\end{align*}
\]