Ch1. Analysis of Algorithms

Algorithms and Data Structures

- An **algorithm** is a step-by-step procedure for performing some task in a finite amount of time.
  - Typically, an algorithm takes input data and produces an output based upon it.

- A **data structure** is a systematic way of organizing and accessing data.
Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition, noting the time needed:
- Plot the results

Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used
Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
  - More structured than English prose
  - Less detailed than a program
- Preferred notation for describing algorithms
- Easy map to primitive operations of CPU

Algorithm arrayMax(A, n):

- **Input**: An array A storing \( n \geq 1 \) integers.
- **Output**: The maximum element in A.

\[
\text{currentMax} \leftarrow A[0] \\
\text{for } i \leftarrow 1 \text{ to } n-1 \text{ do} \\
\quad \text{if } \text{currentMax} < A[i] \text{ then} \\
\quad \quad \text{currentMax} \leftarrow A[i] \\
\text{return } \text{currentMax}
\]
Pseudocode Details

- Control flow
  - if ... then ... [else ...]
  - while ... do ...
  - for ... do ...
  - Indentation replaces braces

- Method declaration
  Algorithm method (arg [, arg...])
  Input ...
  Output ...

- Method call
  method (arg [, arg...])

- Return value
  return expression

- Expressions:
  $\leftarrow$ Assignment
  = Equality testing
  $n^2$ Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

A **RAM** consists of
- A CPU
- An potentially unbounded bank of memory cells, each of which can hold an arbitrary number or character
- Memory cells are numbered and accessing any cell in memory takes unit time
### Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

### Examples:
- Arithmetic operations
- Assigning a value to a variable
- Indexing into an array
- Calling a method
- Returning from a method

### Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant $\approx 1$
  - Logarithmic $\approx \log n$
  - Linear $\approx n$
  - $N$-Log-$N$ $\approx n \log n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
  - Exponential $\approx 2^n$

- In a log-log chart, the slope of the line corresponds to the growth rate

\[ n = 10^x, T(n) = 10^y \Rightarrow x = \log n, y = \log(T(n)) \]
Counting Primitive Operations

- Example: By inspecting the pseudocode, we can determine the minimum and maximum number of primitive operations executed by an algorithm, as a function of the input size.

Algorithm `arrayMax(A, n)`:

- **Input:** An array `A` storing `n ≥ 1` integers.
- **Output:** The maximum element in `A`.

```pseudocode
currentMax ← A[0]
for i ← 1 to n - 1 do
    if currentMax < A[i] then
        currentMax ← A[i]
return currentMax
```

<table>
<thead>
<tr>
<th>How many primitive operations at each line?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>3n-1</td>
</tr>
<tr>
<td>2(n-1)</td>
</tr>
<tr>
<td>0 to 2(n-1)</td>
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<tr>
<td>1</td>
</tr>
</tbody>
</table>

Minimum: `2 + 3n-1 + 2(n-1) + 1 = 5n`

Maximum: `2 + 3n-1 + 4(n-1) + 1 = 7n - 2`
Estimating Running Time

- Algorithm arrayMax executes $7n - 2$ primitive operations in the worst case, $5n$ in the best case.

Define:
  - $a =$ Time taken by the fastest primitive operation
  - $b =$ Time taken by the slowest primitive operation

Let $T(n)$ be worst-case time of arrayMax. Then

$$a(5n) \leq T(n) \leq b(7n - 2)$$

Hence, the running time $T(n)$ is bounded by two linear functions.

Running Time

- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus primarily on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics
Growth Rate of Running Time

- Changing the hardware/software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm arrayMax

Why Growth Rate Matters

<table>
<thead>
<tr>
<th>if runtime is...</th>
<th>time for $n + 1$</th>
<th>time for $2n$</th>
<th>time for $4n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c \lg n$</td>
<td>$c \lg (n + 1)$</td>
<td>$c (\lg n + 1)$</td>
<td>$c(\lg n + 2)$</td>
</tr>
<tr>
<td>$cn$</td>
<td>$c(n + 1)$</td>
<td>$2cn$</td>
<td>$4cn$</td>
</tr>
<tr>
<td>$cn \lg n$</td>
<td>$- c n \lg n + cn$</td>
<td>$2cn \lg n + 2cn$</td>
<td>$4cn \lg n + 4cn$</td>
</tr>
<tr>
<td>$cn^2$</td>
<td>$- c n^2 + 2cn$</td>
<td>$4cn^2$</td>
<td>$16cn^2$</td>
</tr>
<tr>
<td>$cn^3$</td>
<td>$- c n^3 + 3cn^2$</td>
<td>$8cn^3$</td>
<td>$64cn^3$</td>
</tr>
<tr>
<td>$c2^n$</td>
<td>$c2^{n+1}$</td>
<td>$c2^{2n}$</td>
<td>$c2^{4n}$</td>
</tr>
</tbody>
</table>

runtime quadruples when problem size doubles
Analyzing Recursive Algorithms

- Use a function, $T(n)$, to derive a recurrence relation that characterizes the running time of the algorithm in terms of smaller values of $n$.

```
Algorithm recursiveMax(A, n):
    Input: An array A storing $n \geq 1$ integers.
    Output: The maximum element in A.
    if $n = 1$ then
        return $A[0]$
    return max{recursiveMax($A, n-1$), $A[n-1]$}
```

$$T(n) = \begin{cases} 
3 & \text{if } n = 1 \\
T(n-1) + 7 & \text{otherwise}.
\end{cases}$$

Constant Factors

- The growth rate is minimally affected by constant factors or lower-order terms.
- Examples:
  - $10^2n + 10^5$ is a linear function.
  - $10^2n^2 + 10^5n$ is a quadratic function.
Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants $c$ and $n_0$ such that $f(n) \leq cg(n)$ for $n \geq n_0$.
- We also say $g(n)$ is an asymptotic upper bound for $f(n)$.

Example: $2n + 10$ is $O(n)$

$2n + 10 \leq cn$

$(c - 2) n \geq 10$

$n \geq 10/(c - 2)$

Pick $c = 3$ and $n_0 = 10$

Relatives of Big-Oh

- **big-Omega**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c g(n)$ for $n \geq n_0$.

- **big-Theta**
  - $f(n)$ is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c'g(n) \leq f(n) \leq c''g(n)$ for $n \geq n_0$.

**Theorem**: $\Theta$ is an equivalence relation. (reflexive, symmetric, and transitive)
Intuition for Asymptotic Notation

**big-Oh**
- \( f(n) \) is \( O(g(n)) \) if \( f(n) \) is asymptotically less than or equal to \( g(n) \)

**big-Omega**
- \( f(n) \) is \( \Omega(g(n)) \) if \( f(n) \) is asymptotically greater than or equal to \( g(n) \)

**big-Theta**
- \( f(n) \) is \( \Theta(g(n)) \) if \( f(n) \) is asymptotically equal to \( g(n) \)

Example Uses of the Relatives of Big-Oh

- \( 5n^2 \) is \( \Omega(n^2) \)
  - \( f(n) \) is \( \Omega(g(n)) \) if there is a constant \( c > 0 \) and an integer constant \( n_0 \geq 1 \) such that \( f(n) \geq cg(n) \) for \( n \geq n_0 \)
    - let \( c = 5 \) and \( n_0 = 1 \)

- \( 5n^2 \) is \( \Omega(n) \)
  - \( f(n) \) is \( \Omega(g(n)) \) if there is a constant \( c > 0 \) and an integer constant \( n_0 \geq 1 \) such that \( f(n) \geq cg(n) \) for \( n \geq n_0 \)
    - let \( c = 1 \) and \( n_0 = 1 \)

- \( 5n^2 \) is \( \Theta(n^2) \)
  - \( f(n) \) is \( \Theta(g(n)) \) if it is \( \Omega(n^2) \) and \( O(n^2) \). We have already seen the former, for the latter recall that \( f(n) \) is \( O(g(n)) \) if there is a constant \( c > 0 \) and an integer constant \( n_0 \geq 1 \) such that \( f(n) \leq cg(n) \) for \( n \geq n_0 \)
    - Let \( c = 5 \) and \( n_0 = 1 \)
Big-Oh, Big-Theta, Big Omega Rules

- If $f(n)$ is a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say “$2n$ is $O(n)$” instead of “$2n$ is $O(n^2)$”
- Use the simplest expression of the class
  - Say “$3n + 5$ is $O(n)$” instead of “$3n + 5$ is $O(3n)$”

<table>
<thead>
<tr>
<th>$\Theta(n^3)$:</th>
<th>n^3</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>5n^3 + 4n</td>
</tr>
<tr>
<td></td>
<td>105n^3 + 4n^2 + 6n</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta(n^2)$:</th>
<th>n^2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5n^2 + 4n + 6</td>
</tr>
<tr>
<td></td>
<td>n^2 + 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta(\log n)$:</th>
<th>\log n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>\log n^2</td>
</tr>
<tr>
<td></td>
<td>\log (n + n^3)</td>
</tr>
</tbody>
</table>

Examples
**Little oh**

f(n) grows slower than g(n) (or g(n) grows faster than f(n)) if

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0, \]

Notation: \( f(n) = o(g(n)) \)
pronounced "little oh"

**Little omega**

f(n) grows faster than g(n) (or g(n) grows slower than f(n)) if

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty, \]

Notation: \( f(n) = \omega(g(n)) \)
pronounced "little omega"
Relation Summary:

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} \infty & \Rightarrow f(n) = \omega(g(n)) \\ C & \Rightarrow f(n) = \Theta(g(n)) \\ 0 & \Rightarrow f(n) = o(g(n)) \end{cases}
\]

Example: Decide the growth rate of \((\log n)^3n\) and \(n^{n/3}\)

A Case Study in Algorithm Analysis

- Given an array of \(n\) integers, find the subarray, \(A[j..k]\) that maximizes the sum

\[
s_{j..k} = a_j + a_{j+1} + \cdots + a_k = \sum_{i=j}^{k} a_i.
\]

- In addition to being an interview question for testing the thinking skills of job candidates, this maximum subarray problem also has applications in pattern analysis in digitized images.
A First (Slow) Solution

Compute the maximum of every possible subarray summation of the array \( A \) separately.

- The outer loop, for index \( j \), will iterate \( n \) times, its inner loop, for index \( k \), will iterate \( 1 \sim n \) times, and the inner-most loop, for index \( i \), will iterate \( 1 \sim n \) times.
- Thus, the running time of the MaxsubSlow algorithm is \( \mathcal{O}(n^3) \).

An Improved Algorithm

- A more efficient way to calculate these summations is to consider **prefix sums**
  
  \[ S_t = a_1 + a_2 + \cdots + a_t = \sum_{i=1}^{t} a_i \]

- If we are given all such prefix sums (and assuming \( S_0 = 0 \)), we can compute any summation \( s_{j,k} \) in constant time as
  
  \[ s_{j,k} = S_k - S_{j-1} \]

Example:

<table>
<thead>
<tr>
<th>( i = 0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = )</td>
<td>-2</td>
<td>-4</td>
<td>3</td>
<td>-1</td>
<td>5</td>
<td>6</td>
<td>-7</td>
<td>-2</td>
<td>4</td>
<td>-3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( S = )</td>
<td>0</td>
<td>-2</td>
<td>-6</td>
<td>-3</td>
<td>-4</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \text{Max} = S_6 - S_2 = 7 - (-6) = 13. \]
An Improved Algorithm, cont.

- Compute all the prefix sums -- $O(n)$
- Then compute all the subarray sums -- $O(n^2)$

A Linear-Time Algorithm

- Instead of computing prefix sum $S_t = s_{1,t}$, let us compute a maximum suffix sum, $M_t$, which is the maximum of 0 and the maximum $s_{j,t}$ for $j = 1, \ldots, t$.

  $$M_t = \max\left\{0, \max_{j=1,\ldots,t} \{s_{j,t}\}\right\}$$

- If $M_t > 0$, then it is the summation value for a maximum subarray that ends at $t$, and if $M_t = 0$, then we can safely ignore any subarray that ends at $t$.

- If we know all the $M_t$ values, for $t = 1, 2, \ldots, n$, then the solution to the maximum subarray problem would simply be the maximum of all these values.

Algorithm MaxsubFaster($A$):

- Input: An $n$-element array $A$ of numbers, indexed from 1 to $n$.  
- Output: The maximum subarray sum of array $A$.

  $S_0 \leftarrow 0$  // the initial prefix sum
  for $i \leftarrow 1$ to $n$ do
    $S_i \leftarrow S_{i-1} + A[i]$

  $m \leftarrow 0$  // the maximum found so far
  for $j \leftarrow 1$ to $n$ do
    for $k \leftarrow j$ to $n$ do
      $s = S_k - S_{k-1}$
      if $s > m$ then
        $m \leftarrow s$

  return $m$
A Linear-Time Algorithm, cont.

- For \( t \geq 2 \), if we have a maximum subarray that ends at \( t \), and it has a positive sum, then it is either \( A[t] \) or it is made up of the maximum subarray that ends at \( t - 1 \) plus \( A[t] \).
- Also, if taking the value of maximum subarray that ends at \( t - 1 \) and adding \( A[t] \) makes this sum no longer be positive, then \( M_t = 0 \), for there is no subarray that ends at \( t \) with a positive summation.
- So we can define \( M_0 = 0 \) and
  \[
  M_t = \max\{0, M_{t-1} + A[t]\}
  \]

Example:

\[
\begin{array}{cccccccccccc}
    t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
A   & -2 & -4 &  3 & -1 &  5 &  6 & -7 & -2 &  4 & -3 &  2 \\
M   &  0 &  0 &  3 &  2 &  7 & 13 &  6 &  4 &  8 &  5 &  7 \\
\end{array}
\]

\[\text{Max} = M_{11} = 13.\]

Algorithm `MaxsubFastest(A)`:

- **Input:** An \( n \)-element array \( A \) of numbers, indexed from 1 to \( n \).
- **Output:** The maximum subarray sum of array \( A \).

\[
M_0 \leftarrow 0 \quad // \text{the initial prefix maximum}
\]

For \( t \leftarrow 1 \) to \( n \) do

\[
M_t \leftarrow \max\{0, M_{t-1} + A[t]\}
\]

\( m \leftarrow 0 \) // the maximum found so far

For \( t \leftarrow 1 \) to \( n \) do

\[
m \leftarrow \max\{m, M_t\}
\]

return \( m \)

- The `MaxsubFastest` algorithm consists of two loops, which each iterate exactly \( n \) times and take \( O(1) \) time in each iteration. Thus, the total running time of the `MaxsubFastest` algorithm is \( O(n) \).
Math you need to Review

- Summations
- Powers
- Logarithms
- Proof techniques
- Basic probability

**Properties of powers:**
- $a^{(b+c)} = a^b a^c$
- $a^{bc} = (a^b)^c$
- $a^b / a^c = a^{b-c}$
- $b = a^{\log_a b}$
- $b^c = a^{c \log_a b}$

**Properties of logarithms:**
- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b (x/y) = \log_b x - \log_b y$
- $\log_b x^a = a \log_b x$
- $\log_b a = \log_x a / \log_x b$

---

Logarithms

$$\log_b y = x \iff b^x = y \iff b^{\log_b y} = y$$

$$\log nm = \log n + \log m$$

$$\log \frac{n}{m} = \log n - \log m$$

$$\log n^r = r \log n$$

$$\log_a n = \frac{\log_b n}{\log_b a}$$
### Summations

\[ \sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n-1) + f(n) \]

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \]

\[ \sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \]

\[ \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} \text{ for } a > 1 \]

---

### Summations

\[ \sum_{i=1}^{n} \frac{1}{i} = O(\ln n) \]

using Integral of 1/x.

\[ \sum_{i=1}^{n} \log i = O(n \log n) \]

using Stirling’s approximation
The Factorial Function

Definition:

\[ n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \]

Stirling’s approximation:

\[ n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]

or \[ \log(n!) = O(n \log n) \]

How to approve it?

Bounds of Factorial Function

Let \[ \log n! = \sum_{x=1}^{n} \log x, \]

then \[ \int_1^n \log x \, dx < \sum_{x=1}^{n} \log x < \int_0^n \log(x+1) \, dx \]

which gives \[ n \log \left(\frac{n}{e}\right) + 1 \leq \log n! \leq (n+1) \log \left(\frac{n+1}{e}\right) + 1. \]

So \[ e \left(\frac{n}{e}\right)^n \leq n! \leq e \left(\frac{n+1}{e}\right)^{n+1}. \]
Average Case Analysis

- In worst case analysis of time complexity we select the maximum cost among all possible inputs of size n.
- In average case analysis, the running time is taken to be the average time over all inputs of size n.
  - Unfortunately, there are infinite inputs.
  - It is necessary to know the probabilities of all input occurrences.
  - The analysis is in many cases complex and lengthy.

What is the average case of executing “currentMax ← A[i]”?

Algorithm arrayMax(A, n):
  Input: An array A storing n ≥ 1 integers.
  Output: The maximum element in A.

  currentMax ← A[0]
  for i ← 1 to n − 1 do
    if currentMax < A[i] then
      currentMax ← A[i]
  return currentMax

Number of Assignments: the worst case is n. If numbers are randomly distributed, then the average case is 1+1/2 + 1/3 + 1/4 + ... + 1/n = O(log n).

This is because A[i] has only 1/i probability to be the max of A[1], A[2], ..., A[i], under the assumption that all numbers are randomly distributed.
Amortized Analysis

- The **amortized running time** of an operation within a series of operations is the worst-case running time of the series of operations divided by the number of operations.

- Example: A growable array, S. When needing to grow:
  a. Allocate a new array B of larger capacity.
  b. Copy A[i] to B[i], for i = 0, . . . , n − 1, where n is size of A.
  c. Let A = B, that is, we use B as the array now supporting A.

Growable Array Description

- Let add(e) be the operation that adds element e at the end of the array
- When the array is full, we replace the array with a larger one
- But how large should the new array be?
  - Incremental strategy: increase the size by a constant c
  - Doubling strategy: double the size

Algorithm `add(e)`

```plaintext
if n = A.length then
    B ← new array of size ...
    for i ← 0 to n−1 do
        B[i] ← A[i]
    A ← B
    n ← n + 1
    A[n−1] ← e
```
Comparison of the Strategies

- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of $n$ add operations.
- We assume that we start with an empty list represented by a growable array of size 1.
- We call amortized time of an add operation the average time taken by an add operation over the series of operations, i.e., $T(n)/n$.

Incremental Strategy Analysis

- Over $n$ add operations, we replace the array $k = n/c$ times, where $c$ is a constant.
- The total time $T(n)$ of a series of $n$ add operations is proportional to

$$n + c + 2c + 3c + 4c + \ldots + kc = n + c(1 + 2 + 3 + \ldots + k) = n + c(k + 1)/2$$

- Since $c$ is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$.
- Thus, the amortized time of an add operation is $O(n)$. 
Doubling Strategy Analysis: The Aggregate Method

- We replace the array $k = \log_2 n$ times.
- The total time $T(n)$ of a series of $n$ push operations is proportional to:
  
  $n + 1 + 2 + 4 + 8 + \ldots + 2^k = n + 2^{k+1} - 1 = 3n - 1$

- $T(n)$ is $O(n)$.
- The amortized time of an add operation is $O(1)$.

Doubling Strategy Analysis: The Accounting Method

- We view the computer as a coin-operated appliance that requires one cyber-dollar for a constant amount of computing time.
- For this example, we shall pay each add operation 3 cyber-dollars.
  - Set a saving account with $s_0 = 0$ initially.
  - The $i$th operation has a budget cost of $a_i = 3$, which is the amortized cost of each operation.
  - The account value after the $i$th add operation is $s_i = s_{i-1} + a_i - c_i$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>10</th>
<th>11</th>
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<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
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<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Array size</td>
<td>1</td>
<td>2</td>
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<td>8</td>
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<td>$s_i$</td>
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</tr>
</tbody>
</table>
Doubling Strategy Analysis:
The Accounting Method

- We shall pay each add operation 3 cyber-dollars, that is, it will have an amortized $O(1)$ amortized running time.
  - We over-pay each add operation not causing an overflow 2 cyber-dollars.
  - An overflow occurs when the array $A$ has $2^i$ elements.
  - Thus, doubling the size of the array will require $2^i$ cyber-dollars.
  - These cyber-dollars are at the elements stored in cells $2^{i-1}$ through $2^i-1$.