Terminology

• **Experiment**
  - A repeatable procedure that yields one of a given set of outcomes
  - Rolling a die, for example

• **Sample space**
  - The range of outcomes possible
  - For a die, that would be values 1 to 6

• **Event**
  - One of the sample outcomes that occurred
  - If you rolled a 4 on the die, the event is the 4

Probability definition

• The probability of an event occurring is:

\[ p(E) = \frac{|E|}{|S|} \]

  - Where \( E \) is the set of desired events (outcomes) and
  - \( S \) is the set of all possible events (outcomes)
  - Note that \( 0 \leq |E| \leq |S| \)
    - Thus, the probability will always be between 0 and 1
    - An event that will never happen has probability 0
    - An event that will always happen has probability 1
Probability is always a value between 0 and 1

- Something with a probability of 0 will never occur
- Something with a probability of 1 will always occur
- You cannot have a probability outside this range!
- Note that when somebody says it has a "100% probability"
  - That means it has a probability of 1

Dice probability

- What is the probability of getting “snake-eyes” (two 1’s) on two six-sided dice?
  - Probability of getting a 1 on a 6-sided die is 1/6
  - Via product rule, probability of getting two 1’s is the probability of getting a 1 AND the probability of getting a second 1
  - Thus, it’s 1/6 * 1/6 = 1/36
- What is the probability of getting a 7 by rolling two dice?
  - There are six combinations that can yield 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)
  - Thus, |E| = 6, |S| = 36, P(E) = 6/36 = 1/6

The game of poker

- You are given 5 cards (this is 5-card stud poker)
- The goal is to obtain the best hand you can
- The possible poker hands are (in increasing order):
  - No pair
  - One pair (two cards of the same face)
  - Two pair (two sets of two cards of the same face)
  - Three of a kind (three cards of the same face)
  - Straight (all five cards sequentially – ace is either high or low)
  - Flush (all five cards of the same suit)
  - Full house (a three of a kind of one face and a pair of another face)
  - Four of a kind (four cards of the same face)
  - Straight flush (both a straight and a flush)
  - Royal flush (a straight flush that is 10, J, K, Q, A)
Poker probability: royal flush

• What is the chance of getting a royal flush?
  – That’s the cards 10, J, Q, K, and A of the same suit

• There are only 4 possible royal flushes

• Possibilities for 5 cards: C(52,5) = 2,598,960

• Probability = 4/2,598,960 = 0.0000015
  – Or about 1 in 650,000

Poker probability: four of a kind

• What is the chance of getting 4 of a kind when dealt 5 cards?
  – Possibilities for 5 cards: C(52,5) = 2,598,960

• Possible hands that have four of a kind:
  – There are 13 possible four of a kind hands
  – The fifth card can be any of the remaining 48 cards
  – Thus, total possibilities is 13 * 48 = 624

• Probability = 624/2,598,960 = 0.00024
  – Or 1 in 4165

Poker probability: flush

• What is the chance of getting a flush?
  – That’s all 5 cards of the same suit

• We must do ALL of the following:
  – Pick the suit for the flush: C(4,1)
  – Pick the 5 cards in that suit: C(13,5)

• As we must do all of these, we multiply the values out (via the product rule)

• This yields \( \binom{4}{1} \cdot \binom{13}{5} = 5148 \)

• Possibilities for 5 cards: C(52,5) = 2,598,960
• Probability = 5148/2,598,960 = 0.00198
  – Or about 1 in 505
• Note that if you don’t count straight flushes as “flush” or “royal flash”, then the number is really 5108
Poker probability: full house

• What is the chance of getting a full house?
  – That's three cards of one face and two of another face
• We must do ALL of the following:
  – Pick the face for the three of a kind: \( \binom{13}{1} \)
  – Pick the 3 of the 4 cards to be used: \( \binom{4}{3} \)
  – Pick the face for the pair: \( \binom{12}{1} \)
  – Pick the 2 of the 4 cards of the pair: \( \binom{4}{2} \)
• As we must do all of these, we multiply the values out (via the product rule)
  \[
  \left( \frac{13}{1} \right) \left( \frac{3}{1} \right) \left( \frac{4}{1} \right) = 3744
  \]
• Possibilities for 5 cards: \( \binom{52}{5} = 2,598,960 \)
• Probability = \( \frac{3744}{2,598,960} = 0.00144 \)
  – Or about 1 in 694

Inclusion-exclusion principle

• The possible poker hands are (in increasing order):
  – Nothing
  – One pair cannot include two pair, three of a kind, four of a kind, or full house
  – Two pair cannot include three of a kind, four of a kind, or full house
  – Three of a kind cannot include four of a kind or full house
  – Straight cannot include straight flush or royal flush
  – Flush cannot include straight flush or royal flush
  – Full house
  – Four of a kind
  – Straight flush cannot include royal flush
  – Royal flush

Poker probability: three of a kind

• What is the chance of getting a three of a kind?
  – That’s three cards of one face
  – Can’t include a full house or four of a kind
• We must do ALL of the following:
  – Pick the face for the three of a kind: \( \binom{13}{1} \)
  – Pick the 3 of the 4 cards to be used: \( \binom{4}{3} \)
  – Pick the two other cards' face values: \( \binom{12}{2} \)
  – We can’t pick two cards of the same face!
  – Pick the suits for the two other cards: \( \binom{4}{1} \cdot \binom{4}{1} \)
• As we must do all of these, we multiply the values out (via the product rule)
  \[
  \left( \frac{13}{1} \right) \left( \frac{4}{1} \right) \left( \frac{12}{2} \right) \left( \frac{4}{1} \right) \left( \frac{4}{1} \right) = 54912
  \]
• Possibilities for 5 cards: \( \binom{52}{5} = 2,598,960 \)
• Probability = \( \frac{54,912}{2,598,960} = 0.0211 \)
  – Or about 1 in 47
Poker hand odds

- The possible poker hands are (in increasing order):
  - Nothing: 1,302,540, 0.5012
  - One pair: 1,098,240, 0.4226
  - Two pair: 123,552, 0.0475
  - Three of a kind: 54,912, 0.0211
  - Straight: 10,200, 0.00392
  - Flush: 5,108, 0.00197
  - Full house: 3,744, 0.00144
  - Four of a kind: 624, 0.00024
  - Straight flush: 36, 0.0000139
  - Royal flush: 4, 0.0000015

More on probabilities

- Let $E$ be an event in a sample space $S$. The probability of the complement of $E$ is:
  \[ p(\overline{E}) = 1 - p(E) \]

- Recall the probability for getting a royal flush is 0.0000015
  - The probability of not getting a royal flush is 1 - 0.0000015 or 0.9999985
- Recall the probability for getting a four of a kind is 0.00024
  - The probability of not getting a four of a kind is 1 - 0.00024 or 0.99976

Probability of the union of two events

- Let $E_1$ and $E_2$ be events in sample space $S$

- Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

- Consider a Venn diagram dart-board
Probability of the union of two events

\[ P(E_1 \cup E_2) \]

- If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?
- Let \( n \) be the number chosen
  - \( P(2|n) = \frac{50}{100} \) (all the even numbers)
  - \( P(5|n) = \frac{20}{100} \)
  - \( P(2|n) \) and \( P(5|n) = P(10|n) = \frac{10}{100} \)
  - \( P(2|n) \) or \( P(5|n) = P(2|n) + P(5|n) - P(10|n) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5} \)

When is gambling worth it?

- This is a statistical analysis, not a moral/ethical discussion
- What if you gamble $1, and have a \( \frac{1}{2} \) probability to win $10?
  - If you play 100 times, you will win (on average) 50 of those times
    - Each play costs $1, each win yields $10
    - For $100 spent, you win (on average) $500
  - Average win is $5 (or $10 * \( \frac{1}{2} \)) per play for every $1 spent
- What if you gamble $1 and have a 1/100 probability to win $10?
  - If you play 100 times, you will win (on average) 1 of those times
    - Each play costs $1, each win yields $10
    - For $100 spent, you win (on average) $10
  - Average win is $0.10 (or $10 * \( \frac{1}{100} \)) for every $1 spent
- One way to determine if gambling is worth it:
  - probability of winning * payout ≥ amount spent
  - Or (paying) * payout ≥ investment
  - Of course, this is a statistical measure
When is lotto worth it?

- Many older lotto games you have to choose 6 numbers from 1 to 48
  - Total possible choices is $C(48,6) = 12,271,512$
  - Total possible winning numbers is $C(6,6) = 1$
  - Probability of winning is $0.0000000814$
    - Or 1 in 12.3 million
- If you invest $1 per ticket, it is only statistically worth it if the payout is > $12.3 million
  - As, on the “average” you will only make money that way
  - Of course, “average” will require trillions of lotto plays…

Powerball lottery

- Modern powerball lottery is a bit different
  - Source: http://en.wikipedia.org/wiki/Powerball
- You pick 5 numbers from 1-55
  - Total possibilities: $C(55,5) = 3,478,761$
- You then pick one number from 1-42 (the powerball)
  - Total possibilities: $C(42,1) = 42$
- By the product rule, you need to do both
  - So the total possibilities is $3,478,761 \times 42 = 146,107,962$
- While there are many “sub” prizes, the probability for the jackpot is about 1 in 146 million
  - You will “break even” if the jackpot is $146M
  - Thus, one should only play if the jackpot is greater than $146M
- If you count in the other prizes, then you will “break even” if the jackpot is $121M

Blackjack

- You are initially dealt two cards
  - 10, J, Q and K all count as 10
  - Ace is EITHER 1 or 11 (player’s choice)
- You can opt to receive more cards (a “hit”)
- You want to get as close to 21 as you can
  - If you go over, you lose (a “bust”)
- You play against the house
  - If the house has a higher score than you, then you lose
Blackjack probabilities

- Getting 21 on the first two cards is called a blackjack
  - Or a "natural 21"
- Assume there is only 1 deck of cards
- Possible blackjack blackjack hands:
  - First card is an A, second card is a 10, J, Q, or K
    - 4/52 for Ace, 16/51 for the ten card
    - = (4\*16)/(52\*51) = 0.0241 (or about 1 in 41)
  - First card is a 10, J, Q, or K, second card is an A
    - 16/52 for the ten card, 4/51 for Ace
    - = (16\*4)/(52\*51) = 0.0241 (or about 1 in 41)
- Total chance of getting a blackjack is the sum of the two:
  - \( p = 0.0483 \), or about 1 in 20.72

Another way to get 1 in 20.72.

- There are \( C(52,2) = 1,326 \) possible initial blackjack hands
- Possible blackjack blackjack hands:
  - Pick your Ace: \( C(4,1) \)
  - Pick your 10 card: \( C(16,1) \)
  - Total possibilities is the product of the two (64)
- Probability is \( 64/1,326 = 1 \) in 20.72 (0.0483)
**Blackjack probabilities**

- Getting 21 on the first two cards is called a blackjack
- Assume there is an infinite deck of cards
  - So many that the probability of getting a given card is not affected by any cards on the table
- Possible blackjack blackjack hands:
  - First card is an A, second card is a 10, J, Q, or K
    - 4/52 for Ace, 16/52 for second part
    - (4*16)/(52*52) = 0.0236 (or about 1 in 42)
  - First card is a 10, J, Q, or K; second card is an A
    - 16/52 for first part, 4/52 for Ace
    - (16*4)/(52*52) = 0.0236 (or about 1 in 42)
- Total chance of getting a blackjack is the sum:
  - p = 0.0473, or about 1 in 21
  - More specifically, it’s 1 in 21.13 (vs. 20.72)
- In reality, most casinos use “shoes” of 6-8 decks for this reason
  - It slightly lowers the player’s chances of getting a blackjack
  - And prevents people from counting the cards...

**Counting cards and Continuous Shuffling Machines (CSMs)**

- Counting cards means keeping track of which cards have been dealt, and how that modifies the chances
  - There are “easy” ways to do this – count all aces and 10-cards instead of all cards
- Yet another way for casinos to get the upper hand
  - It prevents people from counting the “shoes” of 6-8 decks of cards
- After cards are discarded, they are added to the continuous shuffling machine
- Many blackjack players refuse to play at a casino with one
  - So they aren’t used as much as casinos would like

**So always use a single deck, right?**

- Most people think that a single-deck blackjack table is better, as the player’s odds increase
  - And you can try to count the cards
- But it’s usually not the case!
- Normal rules have a 3:2 payout for a blackjack
  - If you bet $100, you get your $100 back plus 3/2 * $100, or $150 additional
- Most single-deck tables have a 6:5 payout
  - You get your $100 back plus 6/5 * $100 or $120 additional
  - This lowered benefit of awards OUTWEIGHS the benefit of the single deck!
  - You cannot win money on a 6:5 blackjack table that uses 1 deck
  - Remember, the house always wins
Blackjack probabilities: when to hold

- House usually holds on a 17
  - What is the chance of a bust if you draw on a 17? 16? 15?
- Assume all cards have equal probability

- Bust on a draw on a 18
  - 4 or above will bust: that’s 10 (of 13) cards that will bust
    - $10/13 = 0.769$ probability to bust
- Bust on a draw on a 17
  - 5 or above will bust: $9/13 = 0.692$ probability to bust
- Bust on a draw on a 16
  - 6 or above will bust: $8/13 = 0.615$ probability to bust
- Bust on a draw on a 15
  - 7 or above will bust: $7/13 = 0.538$ probability to bust
- Bust on a draw on a 14
  - 8 or above will bust: $6/13 = 0.462$ probability to bust

Why counting cards doesn’t work well…

- If you make two or three mistakes an hour, you lose any advantage
  - And, in fact, cause a disadvantage!
- You lose lots of money learning to count cards
- Then, once you can do so, you are banned from the casinos

So why is Blackjack so popular?

- Although the casino has the upper hand, the odds are much closer to 50-50 than with other games
  - Notable exceptions are games that you are not playing against the house – i.e., poker
    - You pay a fixed amount per hand
Roulette

- A wheel with 38 spots is spun
  - Spots are numbered 1-36, 0, and 00
  - European casinos don't have the 00
- A ball drops into one of the 38 spots
- A bet is placed as to which spot or spots the ball will fall into
  - Money is then paid out if the ball lands in the spot(s) you bet upon

The Roulette table

- Bets can be placed on:
  - A single number 1/38
  - Two numbers 2/38
  - Four numbers 4/38
  - All even numbers 18/38
  - All odd numbers 18/38
  - The first 18 nums 18/38
  - Red numbers 18/38
The Roulette table

- Bets can be placed on:
  - A single number 1/38 36x
  - Two numbers 2/38 18x
  - Four numbers 4/38 9x
  - All even numbers 18/38 2x
  - All odd numbers 18/38 2x
  - The first 18 nums 18/38 2x
  - Red numbers 18/38 2x

Roulette

- It has been proven that no advantageous strategies exist
- Including:
  - Learning the wheel’s biases
    - Casino’s regularly balance their Roulette wheels
  - Using lasers (yes, lasers) to check the wheel’s spin
    - What casino will let you set up a laser inside to beat the house?

Roulette

- It has been proven that no advantageous strategies exist
- Including:
  - Martingale betting strategy
    - Where you double your bet each time (thus making up for all previous losses)
    - It still won’t work!
    - You can’t double your money forever
      - It could easily take 50 times to achieve finally win
      - If you start with $1, then you must put in $1*2^{50} = $1,25,899,906,424,224 to win this way!
      - That’s 1 quadrillion!
    - See http://en.wikipedia.org/wiki/Martingale_(roulette_system) for more info
What’s behind door number three?

• The Monty Hall problem paradox
  – Consider a game show where a prize (a car) is behind one of three doors
  – The other two doors do not have prizes (goats instead)
  – After picking one of the doors, the host (Monty Hall) opens a different door to show you that the door he opened is not the prize
  – Do you change your decision?
• Your initial probability to win (i.e. pick the right door) is 1/3
• What is your chance of winning if you change your choice after Monty opens a wrong door?
• After Monty opens a wrong door, if you change your choice, your chance of winning is 2/3
  – Thus, your chance of winning doubles if you change
  – Huh?

What’s behind door number three?

Player has equal chance to choose one of the doors. There are three cases.

An aside: probability of multiple events

• Assume you have a 5/6 chance for an event to happen
  – Rolling a 1-5 on a die, for example
• What’s the chance of that event happening twice in a row?
• Cases:
  – Event happening neither time: 1/6 * 1/6 = 1/36
  – Event happening first time: 5/6 * 1/6 = 5/36
  – Event happening second time: 1/6 * 5/6 = 5/36
  – Event happening both times: 5/6 * 5/6 = 25/36
• For an event to happen twice, the probability is the product of the individual probabilities
An aside: probability of multiple events

- Assume you have a 5/6 chance for an event to happen
  - Rolling a 1-5 on a die, for example
- What’s the chance of that event happening at least once?
- Cases:
  - Event happening neither time: $1/6 \times 1/6 = 1/36$
  - Event happening first time: $5/6 \times 1/6 = 5/36$
  - Event happening second time: $1/6 \times 5/6 = 5/36$
  - Event happening both times: $5/6 \times 5/6 = 25/36$
- It’s 35/36!
- For an event to happen at least once, it’s 1 minus the probability of it never happening
- Or the complement of it never happening

Probability vs. odds

- Consider an event that has a 1 in 3 chance of happening
- Probability is 0.333
- Which is a 1 in 3 chance
- Or 2:1 odds
  - Meaning if you play it 3 (2+1) times, you will lose 2 times for every 1 time you win
- This, if you have x:y odds, your probability is $y/(x+y)$
  - The y is usually 1, and the x is scaled appropriately
  - For example 2:2:1
    - That probability is $1/(1+2.2) = 1/3.2 = 0.313$
- 1:1 odds means that you will lose as many times as you win