Basics of Counting

22C:19, Chapter 6
Hantao Zhang

The product rule
• Also called the multiplication rule
• If there are \( n_1 \) ways to do task 1, and \( n_2 \) ways to do task 2
  – Then there are \( n_1 n_2 \) ways to do both tasks in sequence
  – This applies when doing the “procedure” is made up of separate tasks
  – We must make one choice AND a second choice

Product rule example
• Sample question
  – There are 18 math majors and 325 CS majors
  – How many ways are there to pick one math major and one CS major?

• Total is \( 18 \times 325 = 5850 \)
Product rule example

How many strings of 4 decimal digits...

a) Do not contain the same digit twice?
   - We want to choose a digit, then another that is not the same, then another...
     - First digit: 10 possibilities
     - Second digit: 9 possibilities (all but first digit)
     - Third digit: 8 possibilities
     - Fourth digit: 7 possibilities
   - Total = $10 \times 9 \times 8 \times 7 = 5040$

b) End with an even digit?
   - First three digits have 10 possibilities
   - Last digit has 5 possibilities
   - Total = $10 \times 10 \times 10 \times 5 = 5000$

The sum rule

- Also called the addition rule
- If there are $n_1$ ways to do task 1, and $n_2$ ways to do task 2
  - If these tasks can be done at the same time, then...
  - Then there are $n_1 + n_2$ ways to do one of the two tasks
  - We must make one choice OR a second choice

Sum rule example

- Sample question
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major or one CS major?

- Total is $18 + 325 = 343$
Sum rule example

How many strings of 4 decimal digits...
• Have exactly three digits that are 9s?
  – The string can have:
    • The non-9 as the first digit
    • OR the non-9 as the second digit
    • OR the non-9 as the third digit
    • OR the non-9 as the fourth digit
  – Thus, we use the sum rule
  – For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
  – Thus, the answer is 9+9+9+9 = 36

Wedding pictures example

• Consider a wedding picture of 6 people
  – There are 10 people, including the bride and groom
a) How many possibilities are there if the bride must be in the picture
  • Product rule: place the bride AND then place the rest of the party
  • First place the bride
    • She can be in one of 6 positions
  • Next, place the other five people via the product rule
    • There are 9 people to choose for the second person, 8 for the third, etc.
    • Total = 9*8*7*6*5 = 15120
  • Product rule yields 6 * 15120 = 90,720 possibilities

Wedding pictures example

• Consider a wedding picture of 6 people
  – There are 10 people, including the bride and groom
b) How many possibilities are there if the bride and groom must both be in the picture
  • Product rule: place the bride/groom AND then place the rest of the party
  • First place the bride and groom
    • She can be in one of 6 positions
    • He can be in one 5 remaining positions
    • Total of 30 possibilities
  • Next, place the other four people via the product rule
    • There are 8 people to choose for the third person, 7 for the fourth, etc.
    • Total = 8*7*6*5 = 1680
  • Product rule yields 30 * 1680 = 50,400 possibilities
Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

c) How many possibilities are there if only one of the bride and groom are in the picture
   - Sum rule: place only the bride
     - Product rule: place the bride AND then place the rest of the party
       - First place the bride
         - She can be in one of 6 positions
       - Next, place the other five people via the product rule
         - There are 8 people to choose for the second person, 7 for the third, etc.
           - We can’t choose the groom!
         - Total = 6 * 8 * 7 * 6 * 5 * 4 = 6720
     - Product rule yields 6 * 6720 = 40,320 possibilities
   - OR place only the groom
     - Same possibilities as for bride: 40,320
   - Sum rule yields 40,320 + 40,320 = 80,640 possibilities

Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

- Alternative means to get the answer

  c) How many possibilities are there if only one of the bride and groom are in the picture
     - Total ways to place the bride (with or without groom): 90,720
       - From part (a)
     - Total ways for both the bride and groom: 50,400
       - From part (b)
     - Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
     - Same number for the groom
     - Total = 40,320 + 40,320 = 80,640

The inclusion-exclusion principle

- When counting the possibilities, we can’t include a given outcome more than once!

  \[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]
  - Let \( A_1 \) have 5 elements, \( A_2 \) have 3 elements, and 1 element be both in \( A_1 \) and \( A_2 \)
  - Total in the union is 5+3-1 = 7, not 8
Inclusion-exclusion example

• How may bit strings of length eight start with 1 or end with 00?
  - Count bit strings that start with 1
    - Rest of bits can be anything: $2^7 = 128$
    - This is $|A_1|$
  - Count bit strings that end with 00
    - Rest of bits can be anything: $2^6 = 64$
    - This is $|A_2|$
  - Count bit strings that both start with 1 and end with 00
    - Rest of the bits can be anything: $2^5 = 32$
    - This is $|A_1 \cap A_2|$
  - Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - Total is $128 + 64 - 32 = 160$

Bit string possibilities

• How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?
  - Consider 5 consecutive 0s first
    - Sum rule: the 5 consecutive 0’s can start at position 1, 2, 3, 4, 5, or 6
      - Starting at position 1
        - Remaining 5 bits can be anything: $2^5 = 32$
      - Starting at position 2
        - First bit must be a 1
        - Remaining 4 bits can be anything: $2^4 = 16$
      - Otherwise, we are including possibilities from the previous case
      - Total = 32 + 16 + 16 + 16 + 16 + 16 = 112
    - The 5 consecutive 1’s follow the same pattern, and have 112 possibilities
    - There are two cases counted twice (that we thus need to exclude):
      - 0000011111 and 1111100000
    - Total = $112 + 112 - 2 = 222$
Tree diagrams

• We can use tree diagrams to enumerate the possible choices

• Once the tree is laid out, the result is the number of (valid) leaves

Tree diagrams example

• Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s

An example closer to home...

• How many ways can Hawkeyes finish the season 10 and 1 after starting with (5,0)?
An example closer to home…

- How many ways can Hawkeyes finish the season 9 and 2 after starting with (5,0)?

Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

- A permutation is an ordered arrangement of the elements of some set S
  - Let S = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a different permutation of S
- An r-permutation is an ordered arrangement of r elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of r-permutations: P(n,r)
  - The poker hand is one of P(52,5) permutations
Permutations

• Number of poker hands (5 cards):
  – \( P(52,5) = 52 \times 51 \times 50 \times 49 \times 48 = 311,875,200 \)
• Number of (initial) blackjack hands (2 cards):
  – \( P(52,2) = 52 \times 51 = 2,652 \)
• \( r \)-permutation notation: \( P(n,r) \)
  – The poker hand is one of \( P(52,5) \) permutations

\[
P(n,r) = \frac{n!}{(n-r)!} = \prod_{i=r+1}^{n} i
\]

### \( r \)-permutations example

• How many ways are there for 5 people in this class to give presentations?

• There are 27 students in the class
  – \( P(27,5) = 27 \times 26 \times 25 \times 24 \times 23 = 9,687,600 \)
  – Note that the order they go in does matter in this example!

### Permutation formula proof

• There are \( n \) ways to choose the first element
  – \( n-1 \) ways to choose the second
  – \( n-2 \) ways to choose the third
  – …
  – \( n-r+1 \) ways to choose the \( r \)th element

• By the product rule, that gives us:
  \( P(n,r) = n(n-1)(n-2)\ldots(n-r+1) \)
Permutations vs. $r$-permutations

- $r$-permutations: Choosing an ordered 5 card hand is $P(52,5)$
  - When people say “permutations”, they almost always mean $r$-permutations
    - But the name can refer to both

- Permutations: Choosing an order for all 52 cards is $P(52,52) = 52!$
  - Thus, $P(n,n) = n!$

Sample question

- How many permutations of {a, b, c, d, e, f, g} end with a?
  - Note that the set has 7 elements

- The last character must be a
  - The rest can be in any order
- Thus, we want a 6-permutation on the set {b, c, d, e, f, g}
- $P(6,6) = 6! = 720$

- Why is it not $P(7,6)$?

Combinations

- What if order doesn’t matter?
- In poker, the following two hands are equivalent:
  - $A\spadesuit, 5\heartsuit, 7\clubsuit, 10\diamondsuit, K\spadesuit$
  - $K\spadesuit, 10\diamondsuit, 7\clubsuit, 5\heartsuit, A\spadesuit$

- The number of $r$-combinations of a set with $n$ elements, where $n$ is non-negative and $0 \leq r \leq n$ is:
  \[ C(n,r) = \frac{n!}{r!(n-r)!} \]
Combinations example

• How many different poker hands are there (5 cards)?

\[
C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{5 \times 4 \times 3 \times 2 \times 1 \times 47!} = 2,598,960
\]

• How many different (initial) blackjack hands are there?

\[
C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52 \times 51}{2 \times 1} = 1,326
\]

Combination formula proof

• Let \( C(52,5) \) be the number of ways to generate unordered poker hands

• The number of ordered poker hands is \( P(52,5) = 311,875,200 \)

• The number of ways to order a single poker hand is \( P(5,5) = 5! = 120 \)

• The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand

• Thus, \( C(52,5) = P(52,5)/P(5,5) \)

Combination formula proof

• Let \( C(n,r) \) be the number of ways to generate unordered combinations

• The number of ordered combinations (i.e. \( r \)-permutations) is \( P(n,r) \)

• The number of ways to order a single one of those \( r \)-permutations \( P(r,r) \)

• The total number of unordered combinations is the total number of ordered combinations (i.e. \( r \)-permutations) divided by the number of ways to order each combination

• Thus, \( C(n,r) = P(n,r)/P(r,r) \)
Combination formula proof

\[ C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!} \]

Bit strings

• How many bit strings of length 10 contain:
  a) exactly four 1’s?
    • Find the positions of the four 1’s
    • Does the order of these positions matter?
      • No!
      • Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
    • Thus, the answer is \( C(10, 4) = 210 \)
  b) at most four 1’s?
    • There can be 0, 1, 2, 3, or 4 occurrences of 1
    • Thus, the answer is:
      • \( C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4) \)
      • \( = 1 + 10 + 45 + 120 + 210 \)
      • \( = 386 \)
  c) at least four 1’s?
    • There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
    • Thus, the answer is:
      • \( C(10, 4) + C(10, 5) + C(10, 6) + C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) \)
      • \( = 210 + 252 + 210 + 120 + 45 + 10 + 1 \)
      • \( = 848 \)
    • Alternative answer: subtract from \( 2^{10} \) the number of strings with 0, 1, 2, or 3 occurrences of 1
  d) an equal number of 1’s and 0’s?
    • Thus, there must be five 0’s and five 1’s
    • Find the positions of the five 1’s
    • Thus, the answer is \( C(10, 5) = 252 \)
Corollary 1

• Let \( n \) and \( r \) be non-negative integers with \( r \leq n \). Then \( C(n,r) = C(n,n-r) \)

• Proof:
  \[
  C(n,r) = \frac{n!}{r!(n-r)!} \\
  C(n,n-r) = \frac{n!}{(n-r)![(n-(n-r))]} = \frac{n!}{r!(n-r)!}
  \]

Corollary example

• There are \( C(52,5) \) ways to pick a 5-card poker hand
• There are \( C(52,47) \) ways to pick a 47-card hand
• \( P(52,5) = 2,598,960 = P(52,47) \)

• When dealing 47 cards, you are picking 5 cards to not deal
  – As opposed to picking 5 cards to deal
  – Again, the order the cards are dealt in does matter

Combinatorial proof

• A combinatorial proof is a proof that uses counting arguments to prove a theorem
  – Rather than some other method such as algebraic techniques

• Essentially, show that both sides of the proof manage to count the same objects

• Most of the questions in this section are phrased as, “find out how many possibilities there are if …”
  – Instead, we could phrase each question as a theorem:
  – “Prove there are \( x \) possibilities if …”
  – The same answer could be modified to be a combinatorial proof to the theorem
Circular seatings

- How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?
- First, place the first person in the north-most chair
  - Only one possibility
- Then place the other 5 people
  - There are $P(5,5) = 5! = 120$ ways to do that
  - By the product rule, we get $1 \times 120 = 120$
- Alternative means to answer this:
  - There are $P(6,6) = 720$ ways to seat the 6 people around the table
  - For each seating, there are 6 "rotations" of the seating
  - Thus, the final answer is $720/6 = 120$

Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is $P(4,4) = 4! = 24$
  - Two horses tie
    - There are $P(4,2) = 6$ ways to choose the two horses that tie
    - There are $P(3,3) = 6$ ways for the "groups" to finish
      - A "group" is either a single horse or the two tying horses
      - By the product rule, there are $6 \times 6 = 36$ possibilities for this case
  - Two groups of two horses tie
    - There are $P(4,2) = 6$ ways to choose the two winning horses
    - The other two horses tie for second place
  - Three horses tie with each other
    - There are $P(4,3) = 4$ ways to choose the two horses that tie
    - There are $P(2,2) = 2$ ways for the "groups" to finish
    - By the product rule, there are $4 \times 2 = 8$ possibilities for this case
  - All four horses tie
    - There is only one combination for this
    - By the sum rule, the total is $24 + 36 + 6 + 8 + 1 = 75$

A last note on combinations

- An alternative (and more common) way to denote an $r$-combination:
  $$C(n,r) = \binom{n}{r}$$
- I’ll use $C(n,r)$ whenever possible, as it is easier to write in PowerPoint