Predicates and Quantifiers

22c:19, Chapter 2
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Terminology review

• Proposition: a statement that is either true or false
  – Must always be one or the other!
  – Example: “The sky is red”
  – Not a proposition: $x + 3 > 4$

• Boolean variable: A variable (usually p, q, r, etc.) that represents a proposition

Propositional functions

• Consider $P(x) = x < 5$
  – $P(x)$ has no truth values ($x$ is not given a value)
  – $P(1)$ is true
    • The proposition $1 < 5$ is true
  – $P(10)$ is false
    • The proposition $10 < 5$ is false

• Thus, $P(x)$ will create a proposition when given a value
Propositional functions

- Let \( P(x) = \text{"x is a multiple of 5"} \)
  - For what values of \( x \) is \( P(x) \) true?

- Let \( P(x) = x+1 > x \)
  - For what values of \( x \) is \( P(x) \) true?

- Let \( P(x) = x + 3 \)
  - For what values of \( x \) is \( P(x) \) true?

Anatomy of a propositional function

\[
P(x) = x + 5 > x
\]

Propositional functions

- Functions with multiple variables:
  - \( P(x,y) = x + y = 0 \)
    - \( P(1,2) \) is false, \( P(1,-1) \) is true
  - \( P(x,y,z) = x + y = z \)
    - \( P(3,4,5) \) is false, \( P(1,2,3) \) is true
  - \( P(x_1,x_2,x_3, \ldots, x_n) = \ldots \)
So, why do we care about quantifiers?

- Many things (in this course and beyond) are specified using quantifiers
  - In some cases, it's a more accurate way to describe things than Boolean propositions

Quantifiers

- A quantifier is "an operator that limits the variables of a proposition"

- Two types:
  - Universal
  - Existential

Universal quantifiers

- Represented by an upside-down A: $\forall$
  - It means “for all”
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\forall x \ P(x)$
  - English translation: “for all values of x, P(x) is true”
  - English translation: “for all values of x, x+1>x is true”
Universal quantifiers

• But is that always true?
  – \( \forall x \ P(x) \)
• Let \( x \) = the character ‘a’
  – Is ‘a’+1 > ‘a’?
• Let \( x \) = the state of Virginia
  – Is Virginia+1 > Virginia?
• You need to specify your universe!
  – What values \( x \) can represent
  – Called the “domain” or “universe of discourse” by the textbook

Universal quantifiers

• Let the universe be the real numbers.
  – Then, \( \forall x \ P(x) \) is true
• Let \( P(x) = x/2 < x \)
  – Not true for the negative numbers!
  – Thus, \( \forall x \ P(x) \) is false
    • When the domain is all the real numbers
• In order to prove that a universal quantification is true, it must be shown for ALL cases
• In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

Universal quantification

• Given some propositional function \( P(x) \)
• And values in the universe \( x_1 .. x_n \)
• The universal quantification \( \forall x \ P(x) \) implies:

\[
P(x_1) \land P(x_2) \land ... \land P(x_n)
\]
Universal quantification

- Think of \( \forall \) as a for loop:
- \( \forall x \ P(x), \) where \( 1 \leq x \leq 10 \)
- \( \ldots \) can be translated as \( \ldots \)
  
  \[
  \text{for ( } x = 1; \ x \leq 10; \ x++ )
  \]

  \[
  \text{is } P(x) \text{ true?}
  \]
- If \( P(x) \) is true for all parts of the for loop, then \( \forall x \ P(x) \)
  
  \[
  \text{Consequently, if } P(x) \text{ is false for any one value of the for loop, then } \forall x \ P(x) \text{ is false}
  \]

Existential quantification

- Represented by an backwards E: \( \exists \)
  - It means “there exists”
  - Let \( P(x) = x+1 > x \)
- We can state the following:
  - \( \exists x \ P(x) \)
  - English translation: “there exists (a value of) \( x \) such that \( P(x) \) is true”
  - English translation: “for at least one value of \( x; x+1>x \) is true”

Existential quantification

- Note that you still have to specify your universe
  - If the universe we are talking about is all the states in the US, then \( \exists x \ P(x) \) is not true
- Let \( P(x) = x+1 < x \)
  - There is no numerical value \( x \) for which \( x+1<x \)
  - Thus, \( \exists x \ P(x) \) is false
Existential quantification

• Let \( P(x) = x + 1 > x \)
  – There is a numerical value for which \( x + 1 \geq x \)
  – In fact, it's true for all of the values of \( x \!
  
• Thus, \( \exists x \; P(x) \) is true

• In order to show an existential quantification is true, you only have to find ONE value
• In order to show an existential quantification is false, you have to show it's false for ALL values

A note on quantifiers

• Recall that \( P(x) \) is a propositional function
  – Let \( P(x) \) be \( "x == 0" \)
• Recall that a proposition is a statement that is either true or false
  – \( P(x) \) is not a proposition
• There are two ways to make a propositional function into a proposition:
  – Supply it with a value
    • For example, \( P(5) \) is false, \( P(0) \) is true
  – Provide a quantification
    • For example, \( \forall x \; P(x) \) is false and \( \exists x \; P(x) \) is true
  – Let the universe of discourse be the real numbers
Binding variables

- Let \( P(x,y) \) be \( x > y \)
- Consider: \( \forall x \ P(x,y) \)
  - This is not a proposition!
  - What is \( y \)?
    - If it's 5, then \( \forall x \ P(x,y) \) is false
    - If it's \( x-1 \), then \( \forall x \ P(x,y) \) is true
- Note that \( y \) is not "bound" by a quantifier

\( (\exists x \ P(x)) \lor Q(x) \)
- The \( x \) in \( Q(x) \) is not bound; thus not a proposition

\( (\exists x \ P(x)) \lor (\forall x \ Q(x)) \)
- Both \( x \) values are bound; thus it is a proposition

\( (\exists x \ P(x) \land Q(x)) \lor (\forall y \ R(y)) \)
- All variables are bound; thus it is a proposition

\( (\exists x \ P(x) \land Q(y)) \lor (\forall y \ R(y)) \)
- The \( y \) in \( Q(y) \) is not bound; this not a proposition

Negating quantifications

- Consider the statement:
  - All students in this class have red hair
- What is required to show the statement is false?
  - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
  - You negate the propositional function
  - AND you change to an existential quantification
  - \( \neg \forall x \ P(x) = \exists x \neg P(x) \)
Negating quantifications

• Consider the statement:
  – There is a student in this class with red hair
• What is required to show the statement is false?
  – All students in this class do not have red hair
• Thus, to negate an existential quantification:
  – You negate the propositional function
  – AND you change to a universal quantification
  \[ \neg \exists x P(x) = \forall x \neg P(x) \]

Translating from English

• Consider "For every student in this class, that student has studied calculus"
• Rephrased: "For every student x in this class, x has studied calculus"
  – Let C(x) be "x has studied calculus"
  – Let S(x) be "x is a student"
• \[ \forall x C(x) \]
  – True if the universe of discourse is all students in this class

Translating from English

• What about if the universe of discourse is all students (or all people?)
  – \[ \forall x (S(x) \land C(x)) \]
    • This is wrong! Why?
  – \[ \forall x (S(x) \rightarrow C(x)) \]
• Another option:
  – Let Q(x, y) be "x has studied y"
  – \[ \forall x (S(x) \rightarrow Q(x, \text{calculus})) \]
Translating from English

• Consider:
  – “Some students have visited Mexico”
  – “Every student in this class has visited Canada or Mexico”

• Let:
  – \( S(x) \) be “\( x \) is a student in this class”
  – \( M(x) \) be “\( x \) has visited Mexico”
  – \( C(x) \) be “\( x \) has visited Canada”

Translating from English

• Consider: “Some students have visited Mexico”
  – Rephrasing: “There exists a student who has visited Mexico”
  
  \[ \exists x M(x) \]
  – True if the universe of discourse is all students
  – What about if the universe of discourse is all people?
  
  \[ \exists x (S(x) \land M(x)) \]
  – When the universe of discourse is all students

Translating from English

• Consider: “Every student in this class has visited Canada or Mexico”
  – When the universe of discourse is all students
  
  \[ \forall x (M(x) \lor C(x)) \]
  – When the universe of discourse is all people
  
  \[ \forall x (S(x) \rightarrow (M(x) \lor C(x))) \]
  – Why isn’t \( \forall x (S(x) \land (M(x) \lor C(x))) \) correct?
Translating from English

- Note that it would be easier to define $V(x, y)$ as "$x$ has visited $y$"
  - $\forall x (S(x) \land V(x, \text{Mexico}))$
  - $\forall x (S(x) \rightarrow (V(x, \text{Mexico}) \lor V(x, \text{Canada}))$

Translating from English

- Translate the statements:
  - "All hummingbirds are richly colored"
  - "No large birds live on honey"
  - "Birds that do not live on honey are dull in color"
  - "Hummingbirds are small"

- Assign our propositional functions
  - Let $P(x)$ be "$x$ is a hummingbird"
  - Let $Q(x)$ be "$x$ is large"
  - Let $R(x)$ be "$x$ lives on honey"
  - Let $S(x)$ be "$x$ is richly colored"

- Let our universe of discourse be all birds

Translating from English

- Our propositional functions
  - Let $P(x)$ be "$x$ is a hummingbird"
  - Let $Q(x)$ be "$x$ is large"
  - Let $R(x)$ be "$x$ lives on honey"
  - Let $S(x)$ be "$x$ is richly colored"

- Translate the statements:
  - "All hummingbirds are richly colored"
    - $\forall x (P(x) \rightarrow S(x))$
  - "No large birds live on honey"
    - $\neg S(x) \lor R(x)$
    - Alternatively: $\forall x (\neg Q(x) \lor \neg R(x))$
  - "Birds that do not live on honey are dull in color"
    - $\forall x (\neg R(x) \rightarrow \neg S(x))$
  - "Hummingbirds are small"
    - $\forall x (P(x) \rightarrow \neg Q(x))$
Prolog

- A programming language using logic!
- Entering facts:
  - instructor (bloomfield, cs202)
  - enrolled (alice, cs202)
  - enrolled (bob, cs202)
- Entering predicates:
  - teaches (P,S) :- instructor (P, C), enrolled (S, C)
- Extracting data
  - ?enrolled (alice, cs202)
    - Result: yes

Prolog

- Extracting data
  - ?enrolled (X, cs202)
    - Result:
      - alice
      - bob
      - claire
- Extracting data
  - ?teaches (X, alice)
    - Result:
      - bloomfield

Multiple quantifiers

- You can have multiple quantifiers on a statement
  - \( \forall x \exists y \, P(x, y) \)
    - “For all x, there exists a y such that P(x, y)”
    - Example: \( \forall x \exists y \, (x+y = 0) \)
  - \( \exists x \forall y \, P(x, y) \)
    - “There exists an x such that for all y P(x, y) is true”
    - Example: \( \exists x \forall y \, (x' y = 0) \)
Order of quantifiers

• \( \exists x \forall y \) and \( \forall x \exists y \) are not equivalent!

• \( \exists x \forall y P(x,y) \)
  - \( P(x,y) = (x+y == 0) \) is false

• \( \forall x \exists y P(x,y) \)
  - \( P(x,y) = (x+y == 0) \) is true

Negating multiple quantifiers

• Recall negation rules for single quantifiers:
  - \( \neg \forall x P(x) = \exists x \neg P(x) \)
  - \( \neg \exists x P(x) = \forall x \neg P(x) \)
  - Essentially, you change the quantifier(s), and negate what it's quantifying

• Examples:
  - \( \neg(\forall x \exists y P(x,y)) \)
    = \( \exists x \neg(\exists y P(x,y)) \)
    = \( \exists x \forall y \neg P(x,y) \)
  - \( \neg(\forall x \exists y z P(x,y,z)) \)
    = \( \exists x \forall y z \neg P(x,y,z) \)
    = \( \exists x \forall y \exists z \neg P(x,y,z) \)

Negating multiple quantifiers

• Consider \( \neg(\forall x \exists y P(x,y)) = \exists x \forall y \neg P(x,y) \)
  - The left side is saying “for all x, there exists a y such that P is true”
  - To disprove it (negate it), you need to show that “there exists an x such that for all y, P is false”

• Consider \( \neg(\exists x \forall y P(x,y)) = \forall x \exists y \neg P(x,y) \)
  - The left side is saying “there exists an x such that for all y, P is true”
  - To disprove it (negate it), you need to show that “for all x, there exists a y such that P is false”
Translating between English and quantifiers

- The product of two negative integers is positive
  \( \forall x \forall y ((x < 0) \land (y < 0) \rightarrow (xy > 0)) \)
  - Why conditional instead of and?
- The average of two positive integers is positive
  \( \forall x \forall y ((x > 0) \land (y > 0) \rightarrow ((x+y)/2 > 0)) \)
- The difference of two negative integers is not necessarily negative
  \( \exists x \exists y ((x < 0) \land (y < 0) \land (x-y \geq 0)) \)
  - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  \( \forall x \forall y (|x+y| \leq |x| + |y|) \)

Translating between English and quantifiers

- \( \exists x \forall y (x+y = y) \)
  - There exists an additive identity for all real numbers
- \( \forall x \forall y (((x > 0) \land (y < 0)) \rightarrow (x-y > 0)) \)
  - A non-negative number minus a negative number is greater than zero
- \( \exists x \exists y (((x \leq 0) \land (y \leq 0)) \land (x-y > 0)) \)
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- \( \forall x \forall y (((x \neq 0) \land (y \neq 0)) \leftrightarrow (xy \neq 0)) \)
  - The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Negation examples

- Rewrite these statements so that the negations only appear within the predicates
  a) \( \neg \exists y \exists x P(x,y) \)
    \( \forall y \exists x \neg P(x,y) \)
    \( \forall x \forall y \neg P(x,y) \)
  b) \( \neg \forall x \exists y P(x,y) \)
    \( \exists x \exists y P(x,y) \)
    \( \exists y \forall x \neg P(x,y) \)
  c) \( \neg \exists y (Q(y) \land \forall x \neg R(x,y)) \)
    \( \forall y \neg Q(y) \lor \forall x \neg R(x,y) \)
    \( \forall y (\neg Q(y) \lor \exists x \neg R(x,y)) \)
    \( \forall y (\neg Q(y) \lor \exists x R(x,y)) \)
Negation examples

- Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) \( \forall x \exists y \forall z T(x,y,z) \)
   - \( \neg (\forall x \exists y \forall z T(x,y,z)) \)
   - \( \neg \forall x \exists y \forall z T(x,y,z) \)
   - \( \exists x \neg \exists y \forall z T(x,y,z) \)
   - \( \exists x \exists y \neg z T(x,y,z) \)

b) \( \forall x \exists y P(x,y) \lor \forall x \exists y Q(x,y) \)
   - \( \neg (\forall x \exists y P(x,y) \lor \forall x \exists y Q(x,y)) \)
   - \( \neg \forall x \exists y P(x,y) \land \neg \forall x \exists y Q(x,y) \)

Rules of inference for the universal quantifier

- Assume that we know that \( \forall x P(x) \) is true
  - Then we can conclude that \( P(c) \) is true
    - Here \( c \) stands for some specific constant
    - This is called "universal instantiation"

- Assume that we know that \( P(c) \) is true for any value of \( c \)
  - Then we can conclude that \( \forall x P(x) \) is true
  - This is called "universal generalization"

Rules of inference for the existential quantifier

- Assume that we know that \( \exists x P(x) \) is true
  - Then we can conclude that \( P(c) \) is true for some value of \( c \)
  - This is called "existential instantiation"

- Assume that we know that \( P(c) \) is true for some value of \( c \)
  - Then we can conclude that \( \exists x P(x) \) is true
  - This is called "existential generalization"
Example of proof

- Given the hypotheses:
  - "Linda, a student in this class, owns a red convertible."
  - "Everybody who owns a red convertible has gotten at least one speeding ticket"
- Can you conclude: “Somebody in this class has gotten a speeding ticket”?

\[
\begin{align*}
C(Linda) & \\
R(Linda) & \\
\forall x (R(x) \rightarrow T(x)) & \\
\exists x (C(x) \land T(x)) & \\
\end{align*}
\]

Thus, we have shown that “Somebody in this class has gotten a speeding ticket”

Example of proof

1. \(\forall x (R(x) \rightarrow T(x))\)  
2. \(R(Linda) \rightarrow T(Linda)\)  
3. \(R(Linda)\)  
4. \(T(Linda)\)  
5. \(C(Linda)\)  
6. \(C(Linda) \land T(Linda)\)  
7. \(\exists x (C(x) \land T(x))\)

Thus, we have shown that “Somebody in this class has gotten a speeding ticket”

Example of proof

- Given the hypotheses:
  - "There is someone in this class who has been to France"
  - "Everyone who goes to France visits the Louvre"
- Can you conclude: “Someone in this class has visited the Louvre”?

\[
\begin{align*}
\exists x (C(x) \land F(x)) & \\
\forall x (F(x) \rightarrow L(x)) & \\
\exists x (C(x) \land L(x)) & \\
\end{align*}
\]
Example of proof

1. \( \exists x (C(x) \land F(x)) \)  
   1st hypothesis
2. \( C(y) \land F(y) \)  
   Existential instantiation using step 1
3. \( F(y) \)  
   Simplification using step 2
4. \( C(y) \)  
   Simplification using step 2
5. \( \forall x (F(x) \rightarrow L(x)) \)  
   2nd hypothesis
6. \( F(y) \rightarrow L(y) \)  
   Universal instantiation using step 5
7. \( L(y) \)  
   Modus ponens using steps 3 & 6
8. \( C(y) \land L(y) \)  
   Conjunction using steps 4 & 7
9. \( \exists x (C(x) \land L(x)) \)  
   Existential generalization using step 8

Thus, we have shown that “Someone in this class has visited the Louvre”