Problem 5.9 (6 pts)
Let \( T = \{ (M) \mid M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w \} \). Show that \( T \) is undecidable.
Assume \( T \) is decidable and that TM \( R \) decides it. \( A_{TM} \) can be reduced to \( T \) as follows:
\( S = \"On input \langle M, w \rangle:\\n1. Create TM \( Q \) as follows:
   \( Q = \"On input x:\n   1) If x does not have the form 01 or 10 reject.
   2) If x has the form 01, then accept.
   3) Else (x has the form 10), run \( M \) on \( w \) and accept if \( M \) accepts \( w \).\"
2. Run \( R \) on \( \langle Q \rangle \)
3. If \( R \) accepts, accept. Otherwise, reject.\"
The constructed machine \( S \) decides \( A_{TM} \), which is known to be undecidable. Therefore, we must conclude that \( T \) is undecidable.

Problem 5.12 (6 pts)
Consider the problem of determining whether a single-tape Turing machine ever writes a blank symbol over a nonblank symbol during the course of its computation on any input string. Formulate this problem as a language and show that it is undecidable.

Let \( A = \{ (M) \mid M \text{ is a TM and } M \text{ writes a blank symbol over a nonblank symbol during the course of its computation on any string} \} \).

Assume \( A \) is decidable and that TM \( R \) decides it. \( A_{TM} \) can be reduced to \( A \) by constructing the TM \( S \):
\( S = \"On input \langle M, w \rangle:\n1. Create a TM \( Q \) based on \( M \) as follows:
   1) For all transitions which write a blank symbol, replace the blank symbol with a new symbol, say \#\, not in \( \Sigma \).
   2) For all transitions which read a blank symbol, add a transition in which the blank symbol is replaced by \#.
   3) For each transition to the accept state, write a nonblank symbol, overwrite with the blank symbol and then go to the accept state
   4) Simulate the modified \( M \) on \( w \).\"
2. Run \( R \) on \( \langle Q \rangle \)
3. If \( R \) accepts, accept. Otherwise, reject.\"
In this situation, \( Q \) simulates \( M \) and will only write the blank symbol iff \( M \) accepts \( w \) because of the new tape symbol \#. Because \( S \) decides \( A_{TM} \), which is known to be undecidable, we conclude that \( A \) is not decidable.

Problem 5.13 (6 pts)
A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is undecidable.

First, we define the decision problem with the language
USELESS$_{TM} = \{(M,q) \mid q \text{ is a useless state in TM } M\}$.

Next, show that USELESS$_{TM}$ is undecidable by reducing $E_{TM}$ to USELESS$_{TM}$.
Suppose that USELESS$_{TM}$ is decidable and that TM $R$ decides it. For any TM $M$ with accept state $q_{accept}$, $q_{accept}$ is useless if and only if $L(M) = \emptyset$. We can use $R$ to check if $q_{accept}$ is a useless state to decide $E_{TM}$. Consider:

$S = \text{"On input } (M), \text{ where } M \text{ is a TM:}\n1. \text{ Run TM } R \text{ on input } (M, q_{accept}), \text{ where } q_{accept} \text{ is the accept state of } M.\n2. \text{ If } R \text{ accepts, accept. Otherwise, reject."}$

However, because we know $E_{TM}$ is undecidable, there cannot exist a TM that decides USELESS$_{TM}$.

**Problem 5.30 (b,c) (6 pts each) (Use Rice’s Theorem)**

Prove the undecidability of the following languages:

**Part b.**
$A = \{(M) \mid M \text{ is a TM and } 1011 \in L(M)\}$.
The property $P$ in this case is $1011 \in L(M)$. It satisfies the two conditions of Rice’s Theorem. First, it is non-trivial because some TMs contain the string 1011 in their language and others do not. Second, it depends only on the language. If two TMs recognize the same language, either both have descriptions in $P$ (because they both accept 1011), or neither do. Thus, Rice’s theorem implies that $P$ is undecidable.

**Part c.**
$ALL_{TM} = \{(M) \mid M \text{ is a TM and } L(M) = \Sigma^*\}$
The property $P$ in this case is $L(M) = \Sigma^*$. This $P$ is non-trivial, because the TM which accepts everything belongs to $ALL_{TM}$ while the TM that rejects everything does not belong to $ALL_{TM}$. $P$ is a property of the language of TM, because for any two given machines, $M_1$ and $M_2$, where $L(M_1) = L(M_2),$

$\langle M_1 \rangle \in ALL_{TM} \iff L(M_1) = \Sigma^* \iff L(M_2) = \Sigma^* \iff \langle M_2 \rangle \in ALL_{TM}$

Consequently, Rice’s theorem implies that $ALL_{TM}$ is undecidable.