How to Prove NP-Completeness

- A problem $B$ is NP-complete if
  - (membership) $B \in \text{NP}$
  - (NP-hard) For all $A \in \text{NP}$, $A \leq_P B$

Theorems

- SAT is NP-complete.
- If $B$ is NP-complete and $B \leq_P C$ then $C$ is NP-hard.
- SAT $\leq_P 3\text{SAT}$.
- $3\text{SAT} \leq_P \text{CLIQUE}$.

Other NP-complete Languages

- VERTEX-COVER is in NP.
- VERTEX-COVER is in NP-hard.
  - Proof1: Using graph properties.
  - Proof2: Reduce $3\text{SAT}$ to VERTEX-COVER.

To show that $C$ is NP-complete, we provide a polynomial time reduction from $3\text{SAT}$ to $C$.

- $\text{VERTEX-COVER} = \{ \langle G, k \rangle \mid G$ is an undirected graph that has a $k$-node vertex cover $\}$.
Proof 1: VERTEX-COVER is NP-hard

Given a graph \( G = (V, E) \), for any \( X \in V \),

- \( X \) is a vertex-cover iff \( V - X \) is an independent set of \( G \).
- \( X \) is a clique of \( G \) iff \( X \) is an independent set of \( \overline{G} \),
  which is the complement of \( G \).

NP-hardness proof:

1. \( \text{CLIQUE} \leq_p \text{INDEPENDENT}: f(\langle G, k \rangle) = \langle \overline{G}, k \rangle \).
2. \( \text{INDEPENDENT} \leq_p \text{VERTEX-COVER}: f(\langle G, k \rangle) = \langle G, |V| - k \rangle \).

Proof 2: VERTEX-COVER is NP-hard

Theorem 7.44 3SAT \( \leq_p \) VERTEX-COVER.

- \( f(\Phi) = \langle G, m + 2n \rangle \), where \( \Phi \) is a set of \( n \) clauses on \( m \) variables.
- \( G = (V, E) \), where
  \( V = \{ x_i, \overline{x}_i | 1 \leq i \leq m \} \cup \{ l_{i,j} | 1 \leq i \leq 3, 1 \leq j \leq n \} \) and
- \( E = \{(l_{1,j}, l_{2,j}), (l_{1,j}, l_{3,j}), (l_{2,j}, l_{3,j}) | 1 \leq j \leq l \} \cup \{(x_i, l_{j,k}) | 1 \leq i \leq m, 1 \leq j \leq 3, 1 \leq k \leq n, \)
  the \( j^{th} \) literal of clause \( k \) is \( x_i \}).

NP Completeness Proofs

Theorem 7.46 HAM-PATH is NP-complete.

To show that \( C \) is NP-complete, we can provide a polynomial time reduction from 3SAT to \( C \).

Theorem 7.55 UHAM-PATH is NP-complete.

Other NP-complete Languages

Suppose \( G \) is a directed graph:

- \( \text{HAM-PATH} = \{ \langle G, s, t \rangle | G \ \text{has a Hamiltonian path from } s \ \text{to } t \} \).
- \( \text{HAM-PATH}_0 = \{ \langle G \rangle | G \ \text{has a Hamiltonian path} \} \).
- \( \text{HAM-CYCLE} = \{ \langle G \rangle | G \ \text{has a Hamiltonian cycle} \} \).

\( G \) can be also an undirected graph (UHAM-PATH, UHAM-PATH0, UHAM-CYLCE).
SUBSET-SUM is NP-complete

Theorem 7.56 $3SAT \leq_P \text{SUBSET-SUM}$. 

$f(\Phi) = \langle X, t \rangle$, where $\Phi$ is a set of $k$ clauses on $l$ variables, $X$ contains $2(k+l)$ numbers of upto $k+l$ digits, and $t$ is a $(k+l)$-digit number whose first $l$ digits are 1’s and the rest $k$ digits are 3’s.

SUBSET-SUM

$\text{SUBSET-SUM} = \{ \langle S, t \rangle \mid S = \{x_1, ..., x_k\} \text{ and for some } \{y_1, ..., y_t\} \subseteq \{x_1, ..., x_k\} \text{ we have } \sum y_i = t \}$. 

PARTITION

$\text{PARTITION} = \{ \langle S \rangle \mid S = \{x_1, ..., x_n\} \text{ and there exists a subset } Y \subset S \text{ such that } \sum(Y) = \sum(S)/2, \} \text{, where } \sum(X) = \sum_{x \in X} x$. 

Theorem PARTITION is NP-complete. 

BIN-PACKING

$\text{BIN-PACKING} = \{ \langle S, t \rangle \mid S = \{x_1, ..., x_k \mid 0 \leq x_i \leq 1 \} \text{ and } S \text{ can be partitioned into } t \text{ subsets, } S = S_1 \cup S_2 \cup \cdots \cup S_t \text{ such that for each } S_i, \sum(S_i) \leq 1 \}$. 

Theorem BIN-PACKING is NP-complete. 

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