Turing Machines as a Computer

- A function \( f : \Sigma^* \rightarrow \Sigma^* \) is computable if some TM \( M \) exists which on every input \( w \) halts with \( f(w) \) on the tape.
- TM for a computable function halts on every input, like Turing deciders.
- Example: All usual arithmetic functions on integers are computable: \( \langle m, n \rangle \mapsto m + n, \langle m, n \rangle \mapsto m \times n, \langle m, n \rangle \mapsto m^n \).
- Using recursion and subroutines.

Mapping Reducibility

- To reduce a problem \( A \) to a problem \( B \) by mapping reducibility means to find a computable function, \( f : A \rightarrow B \) called a reduction, that converts instances of \( A \) into instances of \( B \), such that for any \( w \in \Sigma^* \), \( w \in A \) iff \( f(w) \in B \).
- Computable functions may be transformations of machine descriptions. A computable function \( f \) may take an input \( w \), where \( w \) is an encoding of a TM \( M \), i.e., \( w = \langle M \rangle \), and may return the description of another TM, \( \langle M' \rangle = f(w) \).

More Examples

- \( \lfloor m/n \rfloor \)
- \( \lfloor \sqrt{n} \rfloor \)
- \( \lfloor \log_m(n) \rfloor \)
Application

Proof of the undecidability of PCP contains two mapping reductions:

- $A_{TM} \leq_{m} MP_{PCP}$ by $f_1: A_{TM} \rightarrow MP_{CP}$
- $MP_{CP} \leq_{m} PCP$ by $f_2: MP_{CP} \rightarrow PCP$
- $A_{TM} \leq PCP$ by $f = f_2(f_1((M, w)))$

Some Known Results

- (Theorem 5.22) If $A \leq_{m} B$ and $B$ is decidable, then $A$ is decidable.
- (Corollary 5.23) If $A \leq_{m} B$ and $A$ is undecidable then $B$ is undecidable.
- (Theorem 5.28) If $A \leq_{m} B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.
- (Corollary 5.29) If $A \leq_{m} B$ and $A$ is not Turing-recognizable, then $B$ is not Turing-recognizable.

Intuitive Meaning of Reducibility

Consider two languages $A_{TM}$ and $\overline{A_{TM}}$.

- Intuitively, they are reducible to each other because a solution to either could be used to solve the other by simply reversing the answer.
- However, we know that $\overline{A_{TM}}$ is not mapping reducible to $A_{TM}$ because $A_{TM}$ is Turing recognizable but $\overline{A_{TM}}$ is not.

Problem 5.22

Show that $A$ is Turing-recognizable iff $A \leq_{m} A_{TM}$.