CS:4330 Theory of Computation
Midterm Exam (100 points)

1. (35 points) Decide if L_1 is decidable or not with a formal proof.

    L_1 = { ⟨M⟩ | M is a standard Turing machine, M accepts some $w \in \Sigma^*$ with more than one billion (i.e., 10^{12}) steps. }

Answer: L_1 is undecidable. Suppose L_1 is decidable and there is a TM decider R for L_1: For any TM M, R(⟨M⟩) = accept iff M accepts some $w \in \Sigma^*$ with more than one billion steps. We use R to construct another decider S for E_{TM}, the emptiness problem of Turing machines, as follows:

S = "On ⟨M⟩, a TM

1. Construct ⟨M'⟩ from ⟨M⟩:

    M' = "On x, an input string

    a. Run 10^{12} useless steps;
    b. Simulate M on x and return the result of the simulation."

2. If R(⟨M'⟩) accepts, reject; otherwise accept."

Obviously, L(M') = L(M) and M' takes more than 10^{12} steps to accept any x. If R(M') returns "accept", it means L(M') ≠ {}, thus L(M) ≠ {}. If R(M') returns "reject", then L(M') = L(M) = {}. So S is indeed a decider for E_{TM}. This completes the proof that E_{TM} is reduced to L_1.

2. (35 points) Decide if L_2 is decidable or not with a formal proof.

    L_2 = { ⟨M⟩ | M is a standard Turing machine, M accepts no more than one billion words. }

Answer: L_2 is undecidable. Suppose L_2 is decidable and there is a TM decider R for L_2: For any TM M, R(⟨M⟩) = accept iff $|L(M)| \leq 10^{12}$. We may use R to construct a decider for A_{TM}, that is, we reduce A_{TM} to L_2:

S = "On ⟨M, w⟩, M is a TM and w is a string

1. Construct ⟨M'⟩ from ⟨M, w⟩:

    M' = "On x, an input string

    a. Simulate M on w;
    b. If M accepts w, accept."

2. If R(⟨M'⟩) accepts, reject; otherwise accept."

Note that L(M') = {} if $w \notin L(M)$; L(M') = Σ^* if $w \in L(M)$. If R(M') returns "accept", it means L(M') = {}, thus $w \notin L(M)$. If R(M') returns "reject", then L(M') = Σ^* and $w \in L(M)$. So S is indeed a decider for A_{TM}. This completes the proof that A_{TM} is reduced to L_1.

Note that L_2 is the encoding of a nontrivial property of Turing recognizable languages. By Rice's Theorem, L_2 is not decidable.

3. (30 points) Two known NP-complete problems are Coloring and CNF-SAT:

Coloring = { $\langle G, k \rangle$ | vertices of graph G can be colored by k colors so that the two ends of each edge in G have different colors. }

CNF-SAT = { $\langle \phi \rangle$ | Boolean formula ϕ is in conjunctive normal form and is satisfiable. }

Show with proof details that Coloring can be reduced to CNF-SAT by polynomial reduction.

Answer: We need to provide a polynomial time function f which takes $\langle G, k \rangle$ and produce a Boolean formula ϕ such that $\langle G, k \rangle \in$Coloring iff ϕ is satisfiable. Suppose G = (V, E), where V = $\{v_1, v_2, \ldots, v_n\}$. We use k*n Boolean variables: $p_{i,j}$, $1 \le i \le n$, $1 \le j \le k$. The meaning of $p_{i,j}$ is that $p_{i,j}$ is true iff v_i is assigned color j. Then $\phi = \phi_{unique} \wedge \phi_{consistent}$

ϕ_{unique}: Every vertex receives a unique color:

$$\phi_{unique} = \wedge_{1 \le i \le n} ((p_{i,1} \vee p_{i,2} \vee \ldots \vee p_{i,k}) \wedge_{1 \le a < b \le k} (\neg p_{i,a} \vee \neg p_{i,b}))$$

$\phi_{consistent}$: Every edge has different colors at the two ends:

$$\phi_{consistent} = \wedge_{1 \le j \le k,\ (x, y) \in E} (\neg p_{x,j} \vee \neg p_{y,j}).$$

Now we claim that iff ϕ is satisfiable G can be colored consistently by k colors.

Suppose G can be colored consistently by k colors and for any i, vertex v_i receives color j. We then assign true to $p_{i,j}$ and false to $p_{i,j'}$ for $j' \neq j$. Obviously, ϕ_{unique} is true under this assignment. $\phi_{consistent}$ is also true under this assignment, because for every edge $(x, y) \in E$, x and y receives different colors, so $p_{x,j}$ and $p_{y,j}$ cannot be true at the same time. Thus, ϕ is satisfiable because this assignment makes ϕ to be true.

On the other hand, if ϕ is satisfiable, then there is an assignment which makes ϕ to be true. By ϕ_{unique}, for any i, $p_{i,j}$ is true for a unique j. So we let v_i receives color j. By $\phi_{consistent}$, the two ends of every edges receive different colors. So this coloring is consistent.

Let $f(\langle G, k \rangle) = \phi$, then f can be done in polynomial time of n, where n = |V|. At first, $|E| = O(n^2)$ and $k < n$; otherwise the solution is trivial. ϕ contains kn variables. The number of variables appearing in ϕ_{unique} is $n(k+k(k-1)/2) = O(n^3)$. The number of variables appearing in $\phi_{consistent}$ is $2k|E| = O(n^3)$. So the size of ϕ is $O(n^3)$, and ϕ can be constructed in $O(n^3)$ time. Thus, f is a polynomial time reduction.