Problem 4.31

We are given a CFG G, so we can construct a grammar G' that has all the rules in G. However, there are some exceptions such that we want to delete rules that include A on the left hand side. Also, we want to replace all A's on the right hand side with a new symbol, e.g., a that is not a symbol in G.

If there exists a symbol a in any string in L(G'), that means that A was used for some derivation of some string in L(G).

Here, we can construct a DFA of a regular expression R that repeats symbol in G or symbol a, followed by a, followed by repeating symbol in G or symbol a. Let language L be the intersection of language G' and R; L will be empty if and only if A is not used. We know that the intersection of a CFL and a regular language is a CFL (refer to problem 2.18), so we can construct a decider.

First, construct a DFA for R and a CFG G' as we did above. Then, construct a PDA for L(G') using its CFG. Then, construct PDA for L from PDA of L(G'). Also, construct DFA of L(R); then construct CFG. Finally, we can submit it to the decider E_{CFG}: Our decider rejects if E_{CFG} accepts based on the fact that A was not used, and our decider accepts, otherwise.

Problem 5.9

$$\text{Let } T = \{\langle M \rangle \mid \text{M is a TM that accepts } w^R \text{ whenever it accepts } w\}$$

Suppose that T is decidable and M_T is its decider. We will construct a decider S for A_{TM}

S = "On input $\langle M, w \rangle$ where M is a TM:

1. Create a TM S' as follows.
   On input x:
   1. If x does not have the form ab or ba, reject
   2. If x has the form ab, accept
   3. If x has the form ba, run M on w and accept if M accepts w. Otherwise, we will reject
2. Run M_T on S'. Output what M_T outputs."

$\Rightarrow$ Since S decides A_{TM} and A_{TM} is undecidable, T is undecidable.

Problem 5.12

Let $E = \{\langle M\rangle \mid$ M is a single tape TM which ever writes a blank symbol over a non-blank symbol on any input$\}$

Suppose that E is decidable and M_E is its decider. We will construct a decider S for A_{TM}

S = "On input $\langle M, w\rangle$ where M is a TM:

1. Create a TM S' as follows.

   On any input:

   1. Simulate M on w. Use symbol $\sqcup'$ instead of $\sqcup$ when writing and treat it like $\sqcup$ when reading.
   2. If M accepts, write a true blank symbol $\sqcup$.
2. Run M_E on S'. Output what M_E outputs."

$\Rightarrow$ Since S decides A_{TM} and A_{TM} is undecidable, E is undecidable.

Problem 5.13

$$\text{Let } USELESS_{TM} = \{\langle M, q\rangle \mid \text{q is a useless state in M}\}$$

Suppose that $USELESS_{TM}$ is decidable, and let U be its decider. We will construct a decider E for E_{TM}:

E = "On input $\langle M\rangle$ where M is a TM:

1. Run U on $\langle M$, q$_{accept}\rangle$, where q_{accept} is M's accept state.
2. Output what U outputs."

$\Rightarrow$ Since E_{TM} is undecidable, $USELESS_{TM}$ is also undecidable.

Problem 5.30b)

Let $L_{TM} = \{\langle M \rangle \mid$ M is a TM and 1011 $\in$ L(M)$\}$

Suppose that L_{TM} is decidable and R is a TM that decides L_{TM}. We show that A_{TM} reduces to R

We can use R to construct TM M' that decides A_{TM}
M' = "On input $\langle M, w \rangle$:
1. On input x, simulate M on w. If M accepts w, then accept if x $\in$ 1001.
2. If accepted, 1011 $\in$ L(M'); Otherwise 1011 $\notin$ L(M')."

To decide A_{TM} we do the following:

"On input $\langle M, w \rangle$, construct TM M' defined above.
1. Run R on input $\langle M \rangle$. If R accepts $\langle M' \rangle$ then accept; Otherwise reject."

$\Rightarrow$ Since A_{TM} is undecidable, R cannot exist. Therefore L_{TM} undecidable.

Problem 5.30c)

$$\text{Let } ALL_{TM} = \{\langle M \rangle \mid \text{M is a TM and and L(M)} = \Sigma^*\}$$

Suppose that ALL_{TM} is decidable and R is a TM that decides ALL_{TM}. We show that A_{TM} reduces to R.

We can use R to construct TM M' that decides A_{TM}.

M' = "On input x, simulate M on w, accepting if M accepts w.
1. If M accepts w, then L(M') = Σ^*.
2. If M does not accept w, then L(M') = $\emptyset$"

To decide A_{TM} we do the following:

"On input $\langle M, w \rangle$, construct TM M' defined above.
1. Run R on input $\langle M' \rangle$
2. If R accepts $\langle M' \rangle$ we accept; Otherwise, reject"

$\Rightarrow$ Since A_{TM} is undecidable, R cannot exist. Therefore ALL_{TM} undecidable.