CS 4330

Theory of Computation

Homework 1 Solutions

0.7a) Let R be the relation on $\mathbb{R}$ given by xRy iff $|x-y| \leq 2$

Reflexive: Want to show $\forall x \in \mathbb{R}$, $x \sim x$

$\Rightarrow |x-x| \leq 2$, $\forall x \in \mathbb{R}$

$\Rightarrow 0 \leq 2$, $\forall x \in \mathbb{R}$

$\Rightarrow |x-y| \leq 2$ is reflexive

Symmetric: Want to show $\forall x, y \in \mathbb{R}$, if $x \sim y$ then $y \sim x$

$\Rightarrow$ Since $|x-y| \leq 2$, then $\begin{cases} x-y \leq 2 \Rightarrow y-x \geq -2 & (i) \\ x-y \geq -2 \Rightarrow y-x \leq 2 & (ii) \end{cases}$

$\Rightarrow$ From (i) and (ii) we conclude $|y-x| \leq 2$

$\Rightarrow |x-y| \leq 2$ is symmetric $\forall x, y \in \mathbb{R}$

Not Transitive: Want to show $\forall x, y, z \in \mathbb{R}$ if $x \sim y$ and $y \sim z$, then $x \not\sim z$ using a counter example

$\Rightarrow$ Let $x=9$, $y=7$ and $z=5$

$\Rightarrow |9-7| \leq 2$ and $|7-5| \leq 2$

$\Rightarrow |x-y| \leq 2$ and $|y-z| \leq 2$

$\Rightarrow x \sim y$ and $y \sim z$

$\Rightarrow$ However, $|9-5| = 4$, $4 > 2$

$\Rightarrow x \not\sim z$

$\Rightarrow |x-y| \leq 2$ is not transitive $\forall x, y, z \in \mathbb{R}$

0.7 b) Let R be the relation on $\mathbb{N}$ given by xRy iff $x = ny$, $n \in \mathbb{N}$

Reflexive: Want to show $\forall x \in \mathbb{N}$, $x \sim x$

$\Rightarrow$ $x = nx$

$\Rightarrow$ $\frac{x}{x} = n$

$\Rightarrow$ $n = 1$, $1 \in \mathbb{N}$

$\Rightarrow$ $x = ny$ is Reflexive , $n \in \mathbb{N}$

Not Symmetric: Want to show $\forall x, y \in \mathbb{N}$, if $x \sim y$ then $y \sim x$ is false

$\Rightarrow$ Let $x = 4$ and $y = 2$

$\Rightarrow$ $4 = 2n$

$\Rightarrow$ $\frac{4}{2} = n$

$\Rightarrow$ $n = 2$, $2 \in \mathbb{N}$

$\Rightarrow$ However: $2 = 4n$

$\Rightarrow$ $n = \frac{2}{4} = \frac{1}{2}$, $\frac{1}{2} \notin \mathbb{N}$

$\Rightarrow$ $x = ny$ is not symmetric

Transitive: Want to show $\forall x, y, z \in \mathbb{N}$ if $x \sim y$ and $y \sim z$, then $x \sim z$

$\Rightarrow$ $x = ny$ and $y = mz$, $n, m \in \mathbb{N}$

$\Rightarrow$ $x = n(mz)$
$= (nm)z$

Let $a = nm$, $a \in \mathbb{N}$

$\Rightarrow$ $x = az$

$\Rightarrow$ $x \sim z$

$\Rightarrow$ $x = ny$ is transitive

0.7 c) Let R be the relation on the natural numbers given by xRy iff $xy \geq 1$

<u>Not Reflexive</u>: Want to show $\forall x \in \mathbb{N}$, $x \sim x$ is false

$\Rightarrow$ Let $x = 0$

$\Rightarrow$ $xx = 0 \not\geq 1$

$\Rightarrow$ $xy \geq 1$ is not Reflexive

<u>Symmetric</u>: Want to show $\forall x, y \in \mathbb{N}$, if $x \sim y$ then $y \sim x$

$\Rightarrow$ If $xy \geq 1$ then by associativity $yx \geq 1$ $\forall x, y \in \mathbb{R}$

$\Rightarrow$ $xy \geq 1$ is Symmetric

<u>Transitive</u>: Want to show $\forall x, y, z \in \mathbb{N}$, if $x \sim y$, $y \sim z$ then $x \sim z$

$\Rightarrow$ If $xy \geq 1$ and $yz \geq 1$

$\Rightarrow$ $xz \geq 1$ $\forall x, y, z \in \mathbb{N}$

$\Rightarrow$ $xy \geq 1$ is Transitive

Q.11 a) Want to prove $S(n) = \frac{1}{2}n(n+1)$ where $S(n) = 1+2+\dots+n$

$\Rightarrow$ Prove $\underbrace{1+2+\dots+n}_{S(n)} = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$

Proof

i) Base case: $n = 1$, $S(1) = 1$

$\Rightarrow \quad 1 = \frac{1(1+1)}{2}$

$\Rightarrow \quad 1 = \frac{1(2)}{2}$

$\Rightarrow \quad 1 = 1$

$\Rightarrow$ Since the left hand side is equal to the right hand side $S(1)$ is true.

ii) Induction Hypothesis: Suppose $1+2+\dots+k = \frac{k(k+1)}{2}$ for some $k \in \mathbb{N}$

Induction Step: Need to show that $S(n)$ is true when $n = k+1$

$\Rightarrow$ We need to show $1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2}$

Note: $1+2+\dots+(k+1) = \underbrace{1+2+\dots+k}_{\frac{k(k+1)}{2}} + (k+1)$

$= \frac{k(k+1)}{2} + k+1$ (Using induction hypothesis)

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k^2+k+2k+2}{2}$

$= \frac{k^2+3k+2}{2}$

$= \frac{(k+1)(k+2)}{2}$

$\Rightarrow$ So $S(k+1)$ is true.

$\therefore$ By mathematical induction, $S(n)$ is true $\forall n \in \mathbb{N}$

0.11 b) Want to prove: $C(n) = \frac{1}{4}n^2(n+1)^2$ where $C(n) = 1^3 + 2^3 + \ldots + n^3$

$\Rightarrow$ Prove: $1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$, $\forall n \in \mathbb{N}$

Proof

i) Base Case: $n = 1$, $C(1)$

$\Rightarrow$ $1^3 = \frac{1}{4}(1)^2(1+1)^2$

$\Rightarrow$ $1 = \frac{1}{4}(2)^2$

$\Rightarrow$ $1 = \frac{1}{4}(4)$

$\Rightarrow$ $1 = 1$

$\Rightarrow$ Since the left hand side is equal to the right hand side $C(1)$ is true.

ii) Induction Hypothesis: Suppose $1^3 + 2^3 + \ldots + k^3 = \frac{1}{4}k^2(k+1)^2$ for some $k \in \mathbb{N}$

Induction Step: Need to show that $C(n)$ is true when $n = k+1$

$\Rightarrow$ We need to show: $1^3 + 2^3 + \ldots + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

Note: $1^3 + 2^3 + \ldots + (k+1)^3 = \underbrace{1^3 + 2^3 + \ldots + k^3}_{\frac{1}{4}k^2(k+1)^2} + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ (Using induction hypothesis)

$= \frac{1}{4}k^2(k+1)^2 + \frac{1}{4}(4)(k+1)^3$

$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$\Rightarrow$ So $C(k+1)$ is true

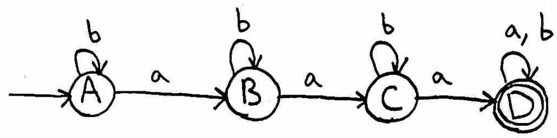
$\therefore$ By mathematical induction, $C(n)$ is true $\forall n \in \mathbb{N}$

1.4a) $L = \{ w \mid w \text{ has at least three a's and at least two b's} \}$

$\Rightarrow L_1 = \{ w \mid w \text{ has at least three a's} \}$

$\Rightarrow L_2 = \{ w \mid w \text{ has at least two b's} \}$

DFA for L_1:



Recall that a DFA is a 5-tuple: $M = (Q, \Sigma, \delta, q_0, F)$

Where: Q is the set of states

Σ is the input alphabet

δ is the transition function

q_0 is the start state

F is the set of final states

1. Set of states: $Q = \{A, B, C, D\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: $\delta : Q \times \Sigma \rightarrow Q$

	a	b
A	B	A
B	C	B
C	D	C
D	D	D

$\Rightarrow \delta(A, a) = B$, $\delta(A, b) = A$

$\Rightarrow \delta(B, a) = C$, $\delta(B, b) = B$

$\Rightarrow \delta(C, a) = D$, $\delta(C, b) = C$

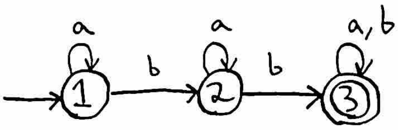
$\Rightarrow \delta(D, a) = D$, $\delta(D, b) = D$

4. Start state: $q_0 = A$

5. Set of final states: $F = \{D\}$

1.4 a) (continued)

DFA for L_2:



1. Set of states: $Q = \{1, 2, 3\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: $\delta: Q \times \Sigma \to Q$

	a	b
1	1	2
2	2	3
3	3	3

$\Rightarrow \delta(1, a) = 1$, $\delta(1, b) = 2$

$\Rightarrow \delta(2, a) = 2$, $\delta(2, b) = 3$

$\Rightarrow \delta(3, a) = 3$, $\delta(3, b) = 3$

4. Start state: $q_0 = 1$
5. Set of final states: $F = \{3\}$

Using footnote on page 46 to construct DFA for L using L_1 and L_2

$\Rightarrow$ 1. Set of states: $Q = \{1A, 1B, 1C, 1D, 2A, 2B, 2C, 2D, 3A, 3B, 3C, 3D\}$

2. Input alphabet: $\Sigma = \{a, b\}$

3. Transition function: For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

$\delta((1,A), a) = (\delta_1(1,a), \delta_2(A,a))$
$= (1, B)$

$\delta((1,A), b) = (\delta_1(1,b), \delta_2(A, b))$
$= (2, A)$

$\delta((1,B), a) = (\delta_1(1,a), \delta_2(B,a))$
$= (1, C)$

$\delta((1,B), b) = (\delta_1(1,b), \delta_2(B,b))$
$= (2, B)$

$\delta((1,C), a) = (\delta_1(1,a), \delta_2(C,a))$
$= (1, D)$

$\delta((1,C), b) = (\delta_1(1,b), \delta_2(C,b))$
$= (2, C)$

$\delta((1,D), a) = (\delta_1(1,a), \delta_2(D,a))$
$= (1, D)$

$\delta((1,D), b) = (\delta_1(1,b), \delta_2(D,b))$
$= (2, D)$

1.4a) (continued)

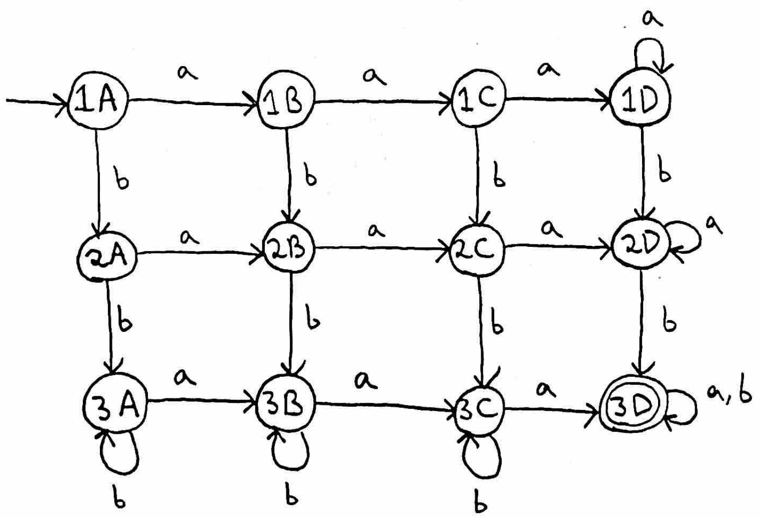
Enumerating all possibilities yields the following transition table:

	a	b
1A	1B	2A
1B	1C	2B
1C	1D	2C
1D	1D	2D
2A	2B	3A
2B	2C	3B
2C	2D	3C
2D	2D	3D
3A	3B	3A
3B	3C	3B
3C	3D	3C
3D	3D	3D

4. Start state: $q_0 = \{1A\}$

5. Set of Final states: $F = \{3D\}$

$\Rightarrow$ DFA for L:

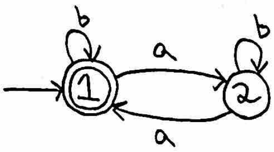


1.4 c) $L = \{ w \mid w \text{ has an even number of } a\text{'s and one or two } b\text{'s} \}$

$\Rightarrow L_1 = \{ w \mid w \text{ has an even number of } a\text{'s} \}$

$\Rightarrow L_2 = \{ w \mid w \text{ has one or two } b\text{'s} \}$

DFA for L_1:



1. Set of states: $Q = \{1, 2\}$

2. Input alphabet: $\Sigma = \{a, b\}$

3. Transition function: $\delta : Q \times \Sigma \to Q$

	a	b
1	2	1
2	1	2

$\Rightarrow \delta(1, a) = 2 \quad , \quad \delta(1, b) = 1$

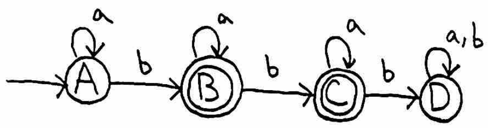
$\Rightarrow \delta(2, a) = 1 \quad , \quad \delta(2, b) = 2$

4. Start state: $q_0 = 1$

5. Set of final states: $F = \{1\}$

1.4 c) (continued)

DFA for L_2:



1. Set of states: $Q = \{A, B, C, D\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: $\delta : Q \times \Sigma \rightarrow Q$

	a	b
A	A	B
B	B	C
C	C	D
D	D	D

$\Rightarrow \delta(A, a) = A$, $\delta(A, b) = B$

$\Rightarrow \delta(B, a) = B$, $\delta(B, b) = C$

$\Rightarrow \delta(C, a) = C$, $\delta(C, b) = D$

$\Rightarrow \delta(D, a) = D$, $\delta(D, b) = D$

4. Start state: $q_0 = A$

5. Set of final states: $F = \{B, C\}$

1.4 c) (continued)

DFA for L using L_1 and L_2: (applying footnote on pg 46)

1. Set of states: $Q = \{1A, 1B, 1C, 1D, 2A, 2B, 2C, 2D\}$

2. Input alphabet: $\Sigma = \{a, b\}$

3. Transition function: For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

$$\delta((1,A), a) = (\delta_1(1,a), \delta_2(A,a)) = (2, A)$$

$$\delta((1,A), b) = (\delta_1(1,b), \delta_2(A,b)) = (1, B)$$

$$\delta((1,B), a) = (\delta_1(1,a), \delta_2(B,a)) = (2, B)$$

$$\delta((1,B), b) = (\delta_1(1,b), \delta_2(B,b)) = (1, C)$$

$$\delta((1,C), a) = (\delta_1(1,a), \delta_2(C,a)) = (2, C)$$

$$\delta((1,C), b) = (\delta_1(1,b), \delta_2(C,b)) = (1, D)$$

$$\delta((1,D), a) = (\delta_1(1,a), \delta_2(D,a)) = (2, D)$$

$$\delta((1,D), b) = (\delta_1(1,b), \delta_2(D,b)) = (1, D)$$

$$\delta((2,A), a) = (\delta_1(2,a), \delta_2(A,a)) = (1, A)$$

$$\delta((2,A), b) = (\delta_1(2,b), \delta_2(A,b)) = (2, B)$$

$$\delta((2,B), a) = (\delta_1(2,a), \delta_2(B,a)) = (1, B)$$

$$\delta((2,B), b) = (\delta_1(2,b), \delta_2(B,b)) = (2, C)$$

$$\delta((2,C), a) = (\delta_1(2,a), \delta_2(C,a)) = (1, C)$$

$$\delta((2,C), b) = (\delta_1(2,b), \delta_2(C,b)) = (2, D)$$

$$\delta((2,D), a) = (\delta_1(2,a), \delta_2(D,a)) = (1, D)$$

$$\delta((2,D), b) = (\delta_1(2,b), \delta_2(D,b)) = (2, D)$$

1.4 c) (continued)

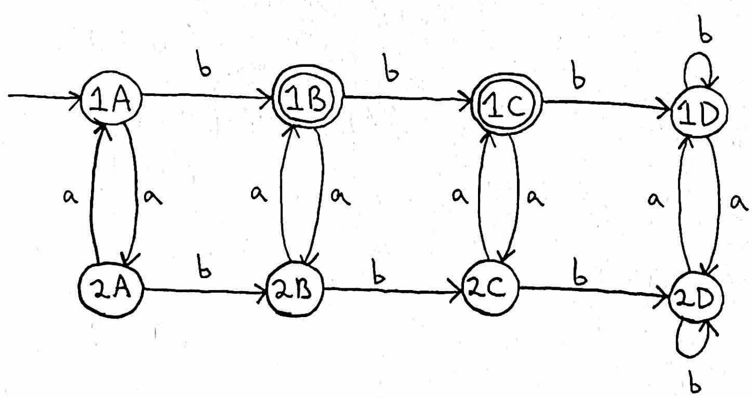
Transition Table for DFA of L:

	a	b
1A	2A	1B
1B	2B	1C
1C	2C	1D
1D	2D	1D
2A	1A	2B
2B	1B	2C
2C	1C	2D
2D	1D	2D

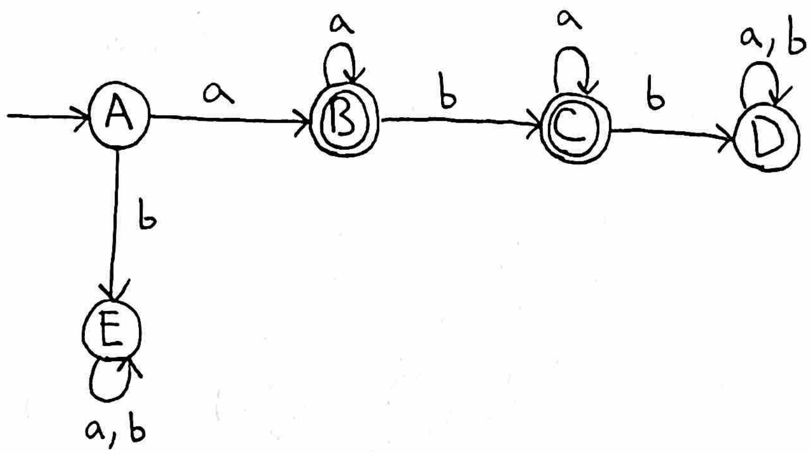
4. Start state: $q_0 = 1A$

5. Set of final states: $F = \{1B, 1C\}$

$\Rightarrow$ DFA for $L = \{ w \mid w$ has an even number of a's and one or two b's$\}$



1.4 e) $L = \{ w \mid w \text{ starts with an a and has at most one b} \}$ (13)

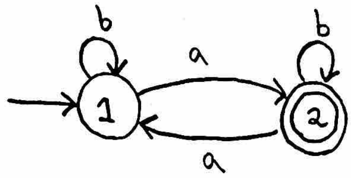


1.4 f) $L = \{ w \mid w \text{ has an odd number of } a\text{'s and ends with a } b \}$

$\Rightarrow L_1 = \{ w \mid w \text{ has an odd number of } a\text{'s} \}$

$\Rightarrow L_2 = \{ w \mid w \text{ ends with a } b \}$

DFA for L_1:



1. Set of states: $Q = \{1, 2\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: $\delta : Q \times \Sigma \to Q$

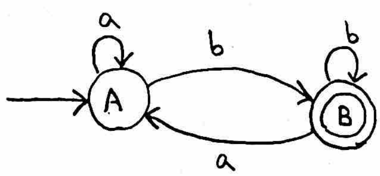
	a	b
1	2	1
2	1	2

$\Rightarrow \delta(1, a) = 2$, $\delta(1, b) = 1$

$\Rightarrow \delta(2, a) = 1$, $\delta(2, b) = 2$

4. Start State: $q_0 = 1$
5. Set of Final States: $F = \{2\}$

DFA for L_2:



1. Set of states: $Q = \{A, B\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: $\delta : Q \times \Sigma \to Q$

	a	b
A	A	B
B	A	B

$\Rightarrow \delta(A, a) = A$, $\delta(A, b) = B$

$\Rightarrow \delta(B, a) = A$, $\delta(B, b) = B$

4. Start state: $q_0 = A$
5. Set of Final States: $F = \{B\}$

DFA for L using L_1 and L_2: (applying footnote on pg 46)

1. Set of states: $Q = \{1A, 1B, 2A, 2B\}$
2. Input alphabet: $\Sigma = \{a, b\}$
3. Transition function: For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

$$\delta((1,A), a) = (\delta_1(1,a), \delta_2(A,a)) = (2, A)$$

$$\delta((1,A), b) = (\delta_1(1,b), \delta_2(A,b)) = (1, B)$$

$$\delta((1,B), a) = (\delta_1(1,a), \delta_2(B,a)) = (2, A)$$

$$\delta((1,B), b) = (\delta_1(1,b), \delta_2(B,b)) = (1, B)$$

$$\delta((2,A), a) = (\delta_1(2,a), \delta_2(A,a)) = (1, A)$$

$$\delta((2,A), b) = (\delta_1(2,b), \delta_2(A,b)) = (2, B)$$

$$\delta((2,B), a) = (\delta_1(2,a), \delta_2(B,a)) = (1, A)$$

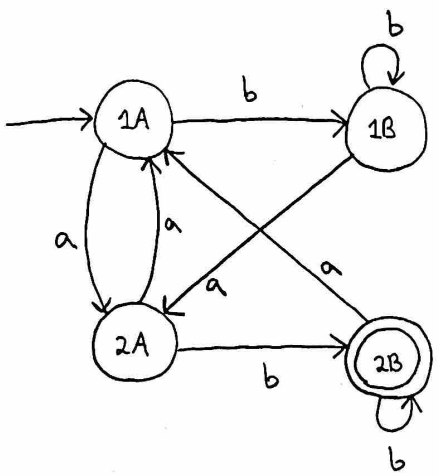
$$\delta((2,B), b) = (\delta_1(2,b), \delta_2(B,b)) = (2, B)$$

Transition Table for L:

	a	b
1A	2A	1B
1B	2A	1B
2A	1A	2B
2B	1A	2B

4. Start state: $q_0 = 1A$
5. Set of Final States: $F = \{2B\}$

DFA for L:

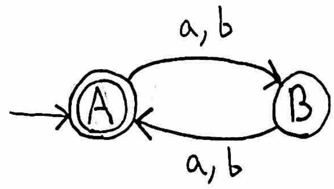


1.4 g) $L = \{ w \mid w \text{ has even length and an odd number of } a\text{'s} \}$

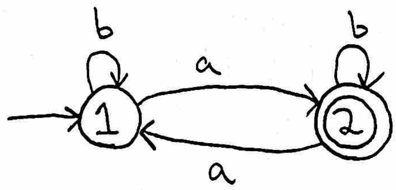
$L_1 = \{ w \mid w \text{ has even length} \}$

$L_2 = \{ w \mid w \text{ has an odd number of } a\text{'s} \}$

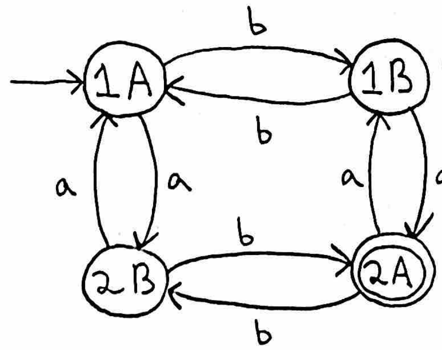
DFA for L_1:



DFA for L_2:



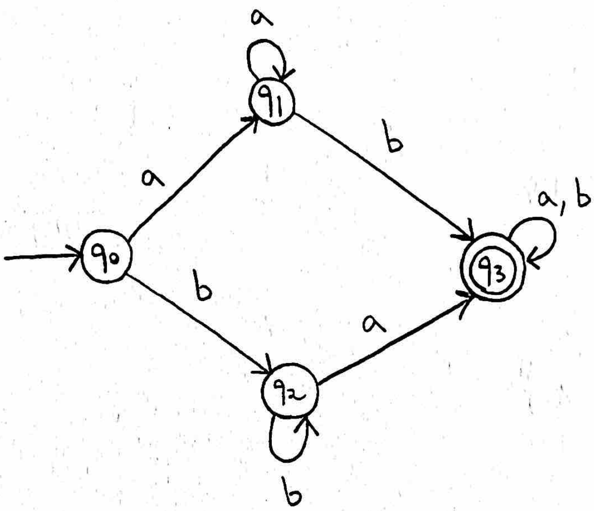
DFA for L:



1.5 c) $L = \{ w \mid w \text{ contains neither the substring } ab \text{ nor } ba \}$

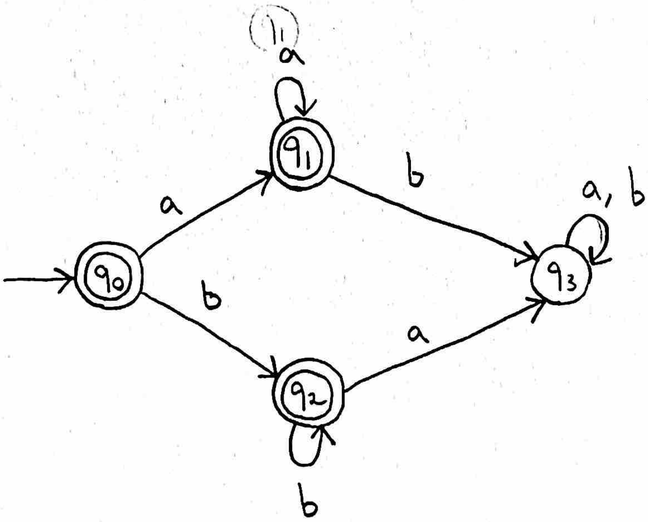
$\Rightarrow \overline{L} = \{ w \mid w \text{ contains either the substring } ab \text{ or } ba \}$

$\Rightarrow$ DFA for $\overline{L}$:



$\Rightarrow$ DFA for L: Accept states in $\overline{L}$ become reject states in L

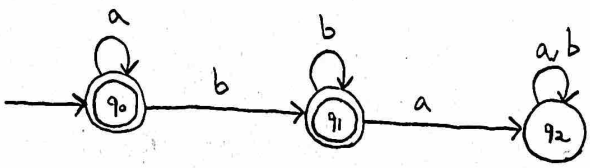
Reject states in $\overline{L}$ become accept states in L



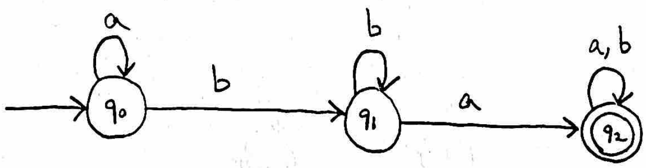
1.5 d) $L = \{ w \mid w \text{ is any string not in } a^*b^* \}$

$\Rightarrow \bar{L} = \{ w \mid w \text{ is any string in } a^*b^* \}$

$\Rightarrow$ DFA for $\bar{L}$:



$\Rightarrow$ DFA for L: Accept states in $\bar{L}$ become reject states in L
Reject states in $\bar{L}$ become reject states in L

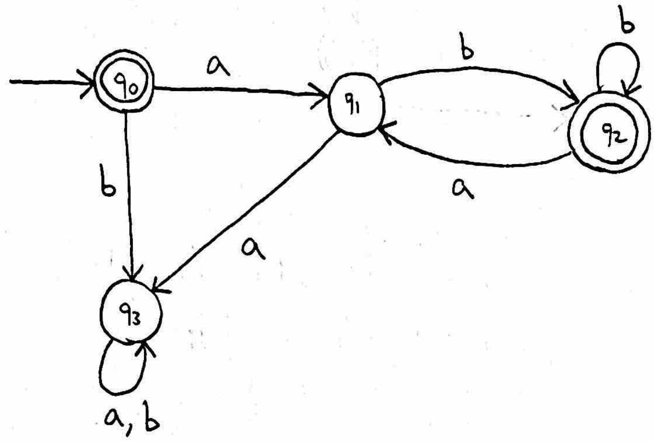


1.5 e)

$L = \{ w \mid w \text{ is any string not in } (ab^+)^* \}$

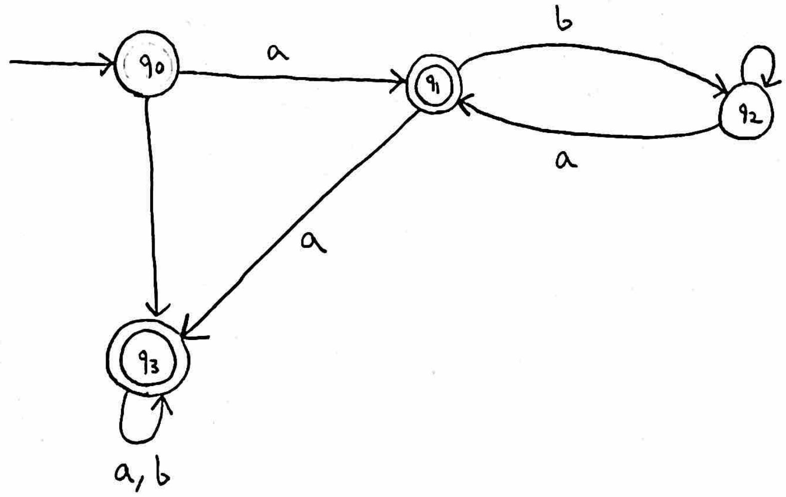
$\Rightarrow \overline{L} = \{ w \mid w \text{ is any string in } (ab^+)^* \}$

$\Rightarrow$ DFA for $\overline{L}$:



$\Rightarrow$ DFA for L: Accept states in $\overline{L}$ become reject states in L

Reject states in $\overline{L}$ become accept states in L

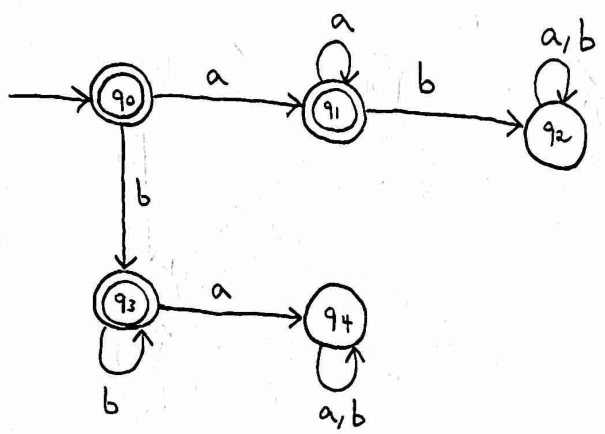


1.5 f)

$L = \{ w \mid w \text{ is any string not in } a^* \cup b^* \}$

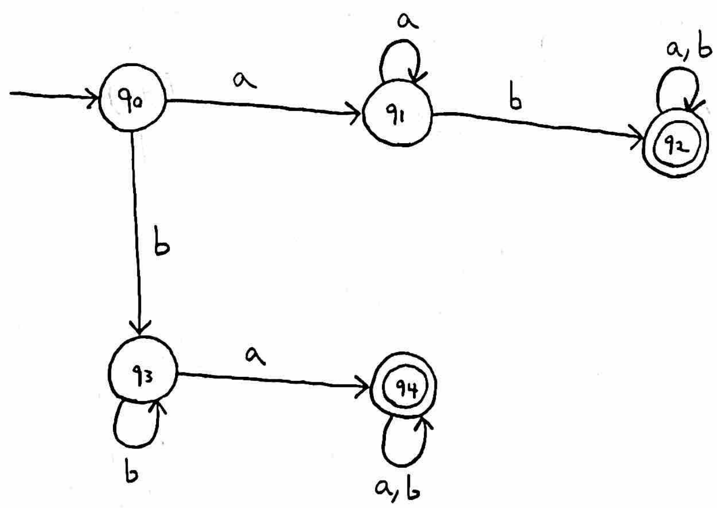
$\Rightarrow \overline{L} = \{ w \mid w \text{ is any string in } a^* \cup b^* \}$

$\Rightarrow$ DFA for $\overline{L}$:



$\Rightarrow$ DFA for L: Accept states in $\overline{L}$ become reject states in L

Reject states in $\overline{L}$ become accept states in L

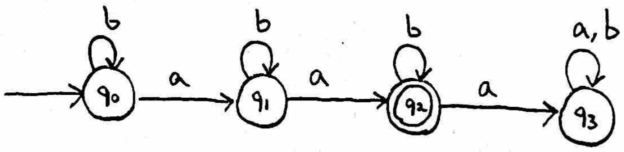


1.5 g)

$L = \{ w \mid w \text{ is any string that doesn't contain exactly two a's} \}$

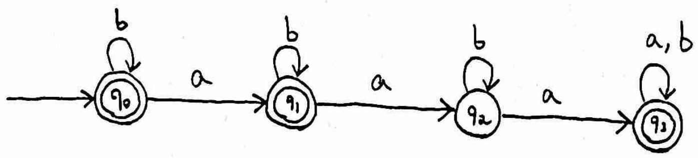
$\Rightarrow \overline{L} = \{ w \mid w \text{ is any string that contains exactly two a's} \}$

$\Rightarrow$ For $\overline{L}$:

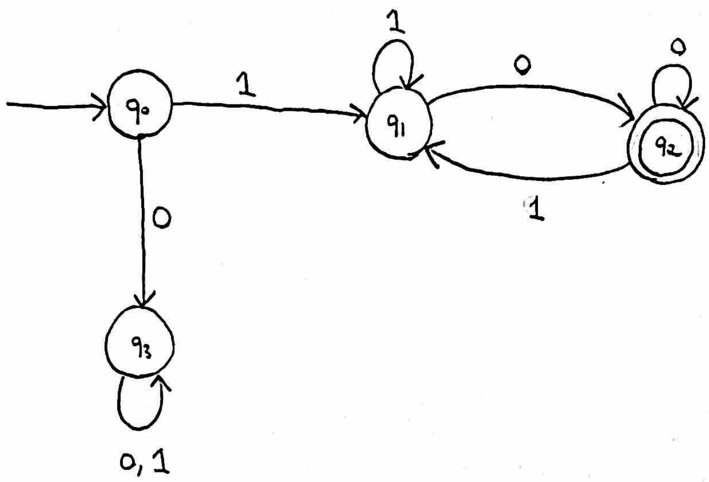


$\Rightarrow$ For L: Accept states in $\overline{L}$ become reject states in L

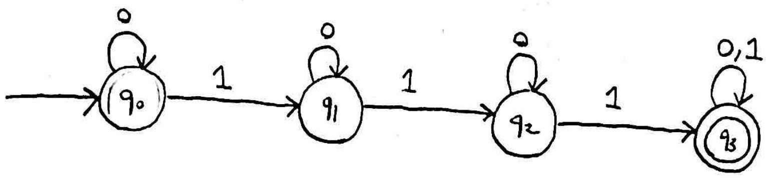
Reject states in $\overline{L}$ become accept states in L



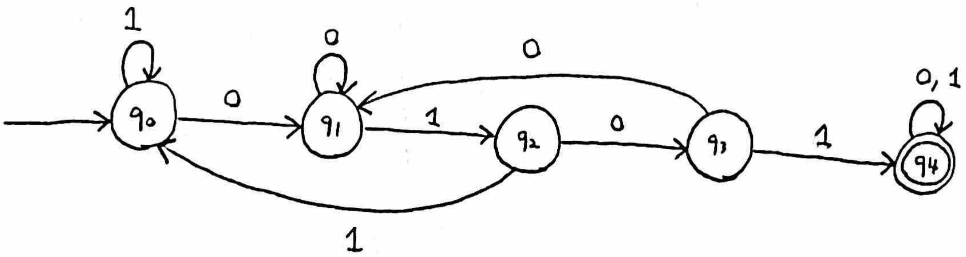
1.6 a) $L = \{ w \mid w \text{ begins with a } 1 \text{ and ends with a } 0 \}$



1.6 b) $L = \{ w \mid w \text{ contains at least three } 1\text{'s} \}$

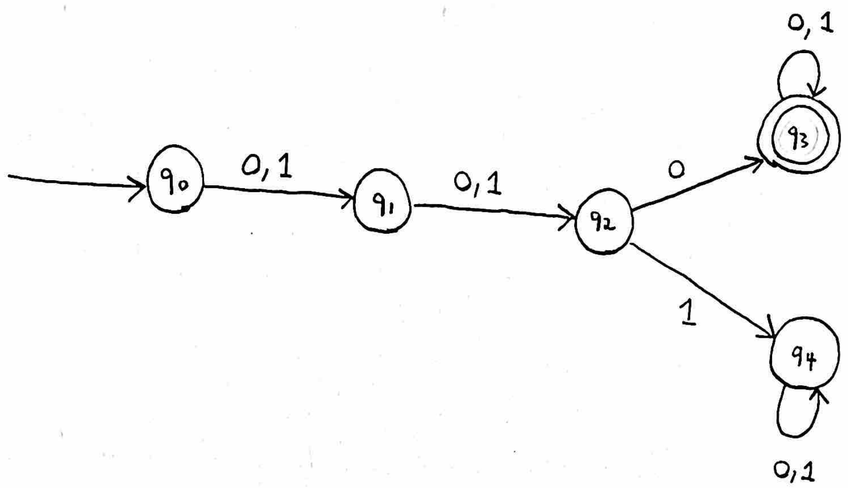


1.6 c) $L = \{ w \mid w \text{ contains the substring } 0101 \}$



1.6 d)

L = { w | w has length at least 3 and its third symbol is a 0 }



1.6 e)

L = { w | w starts with 0 and has odd length, or starts with 1 and has even length }