Computability Theory
Other $PSPACE$-complete problems

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Chapter 8 of [Sipser 1996], 3rd edition. Section 8.3.
A Formula Game

- Consider a quantified Boolean formula $\varphi = \exists x_1 \forall x_2 \exists x_3 \ldots Q x_m. \psi$, with $Q \in \{\exists, \forall\}$, in prenex normal form.

- The game consists of moves by a player $E$ and a player $A$ which take turns assigning values to the variables $x_i$.

- Player $A$ assigns values to the $\forall$ bound variables and player $E$ to the $\exists$ bound variables.

- The order of play is the same as in the quantifier prefix.

- If $\psi$ is true at the end of the game, $E$ wins, otherwise $A$ wins.

- A player is said to have a winning strategy if the player can always win the game by making the right moves, no matter how the other player moves.
Example of Formula Game

Consider the formula

\[ \varphi = \exists x_1 \forall x_2 \exists x_3. [(x_1 \lor x_2) \land (x_2 \lor x_3) \land (\neg x_2 \lor \neg x_3)] \]

▷ In the formula game for \( \varphi \), player \( E \) picks the value for \( x_1 \), then player \( A \) for \( x_2 \) and finally \( E \) picks the value for \( x_3 \)

▷ If \( E \) selects \( x_1 = 1 \), thus \( E \) has the winning strategy by selecting to \( x_3 \) the negation of whatever \( A \) picks to \( x_2 \)

▷ If

\[ \varphi = \exists x_1 \forall x_2 \exists x_3. [(x_1 \lor x_2) \land (x_2 \lor x_3) \land (x_2 \lor \neg x_3)] \]

then \( A \) always wins because no matter how \( E \) assigns \( x_1 \), \( A \) may select \( x_2 = 0 \), which falsifies the last two clauses independently of the value of \( x_3 \)
**Theorem**

The problem of determining which player has a winning strategy in a formula-game associated with a particular formula is \( \text{PSPACE-complete} \), i.e.

\[
\text{FORMULA-GAME} = \left\{ \varphi \mid \text{Player } E \text{ has a winning strategy in the formula game } \varphi \right\}
\]

is \( \text{PSPACE-complete} \).
**Theorem**

The problem of determining which player has a winning strategy in a formula-game associated with a particular formula is PSPACE-complete, i.e. $\text{FORMULA-GAME} = \left\{ \varphi \mid \text{Player } E \text{ has a winning strategy in the formula game } \varphi \right\}$ is PSPACE-complete.

**Proof idea**

Simply show that $\text{FORMULA-GAME} = \text{TQBF}$.

▷ A formula is true exactly when Player $E$ has a winning strategy in the associated formula game.
Generalized Geography

This is a child game where players take turns naming cities from anywhere in the world. Each city chosen must begin with the same letter that ended the previous city name and no duplication is allowed.

Graph model

A directed graph $G = (V, E)$ whose nodes are cities of the world and an arrow goes from node $n_1$ to node $n_2$ if the city labeling $n_2$ starts with the letter that ends the name labeling node $n_1$. 

Theorem

$GG$ is PSPACE-complete.
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$$GG = \left\{ \langle G, b \rangle \mid \text{Player 1 has a winning strategy for the generalized geography game played on graph } G \text{ starting at node } b \right\}$$

Theorem

$GG$ is PSPACE-complete.
The following algorithm $M$ decides whether player 1 has a winning strategy for game $GG$ game:

$M =$ “On input $\langle G, b \rangle$ where $G$ is a directed graph and $b$ is a node of $G$:

1. If $b$ has outdegree 0, reject, player 1 loses immediately
2. Remove node $b$ and all connected arrows to get $G'$
3. For each node $b_1, \ldots, b_k$ that $b$ originally pointed, recursively call $M$ on $\langle G', b_i \rangle$
4. If one of $\langle G', b_i \rangle$ returns “reject”, player 1 would choose $b_i$, and $M$ accepts.
5. If all of these $\langle G', b_i \rangle$ return “accept”, player 2 has a winning strategy in the original game, so $M$ returns reject.”
GG is PSPACE-hard

**Theorem**

**FORMULA-GAME \( \leq_p GG \).**

▷ Let \( \varphi \) be a formula game, where the body of \( \varphi \) is a CNF containing \( m \) clauses. We construct a graph \( G = (V, E) \) and a special node \( b \) such that player \( E \) has a winning strategy in the formula game \( \varphi \) iff player 1 has a winning strategy in \((G, b)\) in GG.

▷ \( V = V_1 \cup V_2 \), where
  - \( V_1 = \{b_i, x_i, \bar{x}_i, e_i \mid 1 \leq i \leq k, k = |X|\} \) (assume \( |X| \) is odd), and
  - \( V_2 = \{c, c_j, c_j,i \mid 1 \leq j \leq k, c_j = (c_{j,1} \lor \cdots \lor c_{j,k})\} \)

▷ \( E = E_1 \cup E_2 \cup E_3 \), where
  - \( E_1 \) constructs a chain of “diamonds” among \( V_1 \)
  - \( E_2 \) constructs a tree among \( V_2 \)
  - \( E_3 \) connects \( V_1 \) and \( V_2 \)

▷ \( b = b_1 \)