Readings for this lecture

Chapter 1 of [Sipser 1996], 3rd edition. Section 1.4.
Consider the language $B = \{0^n1^n \mid n \geq 0\}$

- If we attempt to find a DFA that recognizes $B$ we discover that such a machine needs to remember how many 0s have been seen so far as it reads the input.

- Because the number of 0s isn’t limited, the machine needs to keep track of an unlimited number of possibilities.

- This cannot be done with any finite number of states.
Intuition may fail us

- Just because a language appears to require unbounded memory in order to be recognized, it doesn’t mean that it is necessarily so

Example

- $L_1 = \{w \mid w$ has an equal number of 0s and 1s$\}$ is not regular
- $L_2 = \{w \mid w$ has an equal number of ‘01’s and ‘10’s$\}$ is regular
Language nonregularity

- The technique for proving nonregularity of some languages is provided by a theorem about regular languages called *pumping lemma*.

- The pumping lemma states that all regular languages have a special property.

- If we can show that a language does not have this property we are guaranteed that it is not regular.
Pumping property

All strings in the language can be “pumped” if they are at least as long as a certain special value, called the pumping length

**Meaning:** each such string in the language contains a section that can be repeated any number of times with the resulting string remaining in the language
Pumping lemma

**Theorem**

If $A$ is a regular language, then there is a number $p$ (the pumping length) such that, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the conditions:

1. for each $i \geq 0$, $xy^iz \in A$
2. $|y| > 0$
3. $|xy| \leq p$
Proof idea

Let \( M = (Q, \Sigma, \delta, q_0, F) \) be a DFA that recognizes \( A \)

- Assign a pumping length \( p \) to the number of states of \( M \)
- Show that any string \( s \in A, |s| \geq p \) may be broken into three pieces \( xyz \) satisfying the pumping lemma's conditions

- Note that if there are no strings in \( A \) of length at least \( p \) then the theorem becomes \textit{vacuously} true, since all three conditions hold for all strings of length at least \( p \) if there are no such strings
Pumping lemma’s proof

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing $A$ and $p$ be the number of states of $M$. Let $s = a_1 \ldots a_n$ be a string over $\Sigma$ with $n \geq p$ and $r_1, \ldots, r_{n+1}$ be the sequence of states while processing $s$, i.e. $r_{i+1} = \delta(r_i, a_i)$, $1 \leq i \leq n$

\hspace{1cm} $\Rightarrow$ $n + 1 \geq p + 1$ and among the first $p + 1$ elements in $r_1, \ldots, r_{n+1}$ two must be the same state, say $r_j$ and $r_k$

\hspace{1cm} $\Rightarrow$ Since $r_k$ occurs among the first $p + 1$ places in the sequence starting at $r_1$, we have $k \leq p + 1$

\hspace{1cm} $\Rightarrow$ Now let $x = a_1 \ldots a_{j-1}$, $y = a_j \ldots a_{k-1}$, $z = a_k \ldots a_n$
Note

- As $x$ takes $M$ from $r_1$ to $r_j$, $y$ takes $M$ from $r_j$ to $r_k$ and $z$ takes $M$ from $r_k$ to $r_{n+1}$, which is an accept state, therefore $M$ must accept $xy^i z$, for $i \geq 0$

- We know that $j \neq k$, so $|y| > 0$

- We also know that $k \leq p + 1$, so $|xy| \leq p$

Thus, all conditions are satisfied and the lemma is proven. □
Prove that a language $A$ is not regular using the pumping lemma:

1. Assume that $A$ is regular in order to obtain a contradiction

2. The pumping lemma guarantees the existence of a pumping length $p$ such that all strings of length $p$ or greater in $A$ can be pumped

3. Find $s \in A$, $|s| \geq p$, that cannot be pumped: demonstrate that $s$ cannot be pumped by considering all ways of dividing $s$ into substrings $x, y, z$, showing that for each such division at least one of the conditions of the pumping lemma fails:

   a) $xy^iz \notin A$, for some $i \geq 0$; or
   b) $|y| \leq 0$; or
   c) $|xy| > p$
Applications

Prove that $B = \{0^n1^n \mid n \geq 0\}$ is not regular
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Proof

Assume that $B$ is regular and let $p$ be the pumping length of $B$. Choose $s = 0^p1^p \in B$; therefore $|0^p1^p| > p$. By pumping lemma, $s = xyz$ such that for any $i \geq 0$, $xy^iz \in B$.
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Consider the cases:

1. $y$ consists of $0$s only. Then for some $i$, $xy^iz$ has more $0$s than $1$s and is not in $B$, violating condition (1).
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The contraction is unavoidable if we make the assumption that $B$ is regular, therefore $B$ is not regular.
Example

Prove that $C = \{ w \mid w$ has an equal number of 0s and 1s$\}$ is not regular

Proof

Assume that $C$ is regular and $p$ is its pumping length. Let $s = 0^p 1^p$, hence $s \in C$. Then pumping lemma guarantees that $s = xyz$, where $xy^iz \in C$ for any $i \geq 0$.

If we take the division $x = z = \epsilon$, $y = 0^p 1^p$ it is the case that $xy^iz \in C$, for $i \geq 0$. 
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$\triangleright$ Condition (3) states that $|xy| \leq p$, and in our case $|y| > p$. So $0^p 1^p$ cannot be pumped.
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- Condition (3) states that $|xy| \leq p$, and in our case $|y| > p$. So $0^p 1^p$ cannot be pumped.
- If $|xy| \leq p$, then $y$ must consist of only $0$s.
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\( \triangleright \) Therefore there always is an \( i \) s.t. \( xy^i z \not\in C \), since there are more 0s than 1s. This gives us the desired contradiction.
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If we take the division $x = z = \epsilon$, $y = 0^p 1^p$ it is the case that $xy^iz \in C$, for $i \geq 0$. However:

- Condition (3) states that $|xy| \leq p$, and in our case $|y| > p$. So $0^p 1^p$ cannot be pumped.

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  Therefore there always is an $i$ s.t. $xy^iz \notin C$, since there are more 0s than 1s. This gives us the desired contradiction.

Note that selecting $s = (01)^p$ could have misled us since this string can be pumped by the division: $x = \epsilon$, $y = 01$, $z = (01)^{p-1}$. Then $xy^iz \in C$ for any $i \geq 0$. 
An alternative method for the previous example

Exploit the properties of regular operations.

▷ If $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$ were regular, then $C \cap 0^*1^*$ would also be regular because $0^*1^*$ is regular and the intersection of regular languages is also a regular language.
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- But we know that $B = \{0^n1^n \mid n \geq 0\}$ is not regular.

- Since $C \cap 0^*1^* = B$ is not regular, then $C$ is not regular either.
Another example

Prove that \( F = \{ww \mid w \in \{0, 1\}^*\} \) is nonregular using pumping lemma

Proof idea

Assume that \( F \) is regular and \( p \) is its pumping length. Consider \( s = 0^p10^p1 \in F \). Since \(|s| > p\), it can be divided into some \( s = xyz \) satisfying the conditions.
Another example

Prove that $F = \{ww \mid w \in \{0, 1\}^*\}$ is nonregular using pumping lemma.

**Proof idea**

Assume that $F$ is regular and $p$ is its pumping length. Consider $s = 0^p10^p1 \in F$. Since $|s| > p$, it can be divided into some $s = xyz$ satisfying the conditions. However, it is impossible to satisfy condition (3) unless $y$ consists only of $0s$, and this implies however that $xyyz \not\in F$, violating condition (1). Therefore $F$ is nonregular.
Another example

Prove that $F = \{ww \mid w \in \{0, 1\}^*\}$ is nonregular using pumping lemma

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<th>Proof idea</th>
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Choosing $s = 0^p 10^p 1$ as the candidate for a counterexample was crucial. Taking for example $0^p 0^p$ would lead us no way, since it can be pumped.
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We illustrate this using the pumping lemma to prove that $E = \{0^i1^j \mid i > j\}$ is not regular.
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- **Proof:** by contradiction using pumping lemma. Assume that $E$ is regular and its pumping length is $p$. Let $s = 0^{p+1}1^p$; from decomposition, $s = xyz$. 


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- To satisfy condition (3), i.e. $|xy| \leq p$, then $y$ consists only of 0s.
Pumping down

Sometimes “pumping down” is useful when apply the pumping lemma:

▶ We illustrate this using the pumping lemma to prove that \( E = \{0^i1^j \mid i > j \} \) is not regular

▶ **Proof:** by contradiction using pumping lemma. Assume that \( E \) is regular and its pumping length is \( p \). Let \( s = 0^{p+1}1^p \); from decomposition, \( s = xyz \).

▶ To satisfy condition (3), i.e. \(|xy| \leq p\), then \( y \) consists only of 0s.

▶ Let us examine \( xyyz \) to see if it is in \( E \). Adding an extra-copy of \( y \) increases the \( n \) number of 0s. Since \( E \) contains all strings \( 0^*1^* \) that have more 0s than 1s, this will still give a string in \( E \).
Try something else

Since $x y^i z \in E$ even when $i = 0$, consider $i = 0$ and $x y^0 z = x z \in E$.

This decreases the number of $0$ in $s$.
Since $xy^iz \in E$ even when $i = 0$, consider $i = 0$ and $xy^0z = xz \in E$.

This decreases the number of 0 in $s$.

Since $s$ has just one more 0 than 1 and $|y| \neq 0$, then $xy$ can have at most the same number of 0s and 1s, which means that $xz$ cannot be in $E$.

This provides us the required contradiction.