5.5. Additional Exercises for Chapter 5

Exercise 5.5.1. Let G be a finite group and let H be a subgroup. Let Y denote the set of conjugates of H in G, $Y = \{gHg^{-1} : g \in G\}$. As usual, G/H denotes the set of left cosets of H in G, $G/H = \{gH : g \in G\}$.

(a) Show that $\dfrac{\#(G/H)}{\#Y} = [N_G(H) : H]$.

(b) One has a map from G/H to Y defined by $gH \mapsto gHg^{-1}$. Show that this map is well-defined and surjective.

(c) The map in part (b) is one-to-one if, and only if, $H = N_G(H)$. In general the map is $[N_G(H) : H]$-to-one; i.e. the preimage of each element of Y has size $[N_G(H) : H]$.

Definition 5.5.1. Suppose a group G acts on sets X and Y. One says that a map $\varphi : X \to Y$ is *G-equivariant* if for all $x \in X$,

$$\varphi(g \cdot x) = g \cdot (\varphi(x)).$$

Exercise 5.5.2. Let G act transitively on a set X. Fix $x_0 \in X$, let $H = \mathrm{Stab}(x_0)$, and let Y denote the set of conjugates of H in G. Show that there is a G-equivariant surjective map from X to Y given by $x \mapsto \mathrm{Stab}(x)$, and this map is $[N_G(H) : H]$-to-one.

Exercise 5.5.3. Let $D_4 \subseteq S_4$ be the subgroup generated by (1234) and $(14)(23)$. Show that $N_{S_4}(D_4) = D_4$. Conclude that there is an S_4-equivariant bijection from S_4/D_4 onto the set of conjugates of D_4 in S_4.

Exercise 5.5.4. Let G be the rotation group of the tetrahedron, acting on the set of faces of the tetrahedron. Show that map $F \mapsto \mathrm{Stab}(F)$ is bijective, from the set of faces to the set of stablizer subgroups of faces.

Exercise 5.5.5. Let G be the rotation group of the cube, acting on the set of faces of the cube Show that map $F \mapsto \mathrm{Stab}(F)$ is 2-to-1, from the set of faces to the set of stabilizer subgroups of faces.

Exercise 5.5.6. Let $G = S_n$ and $H = \mathrm{Stab}(n) \cong S_{n-1}$. Show that H is its own normalizer, so that the cosets of H correspond 1-to-1 with conjugates of H. Describe the conjugates of H explicitly.

Exercise 5.5.7. Identify the group G of rotations of the cube with S_4, via the action on the diagonals of the cube. G also acts transitively on the set of set of three four-fold rotation axes of the cube; this gives a homomorphism of S_4 into S_3.

(a) Compute the resulting homomorphism ψ of S_4 to S_3 explicitly. (For example, compute the image of a set of generators of S_4.) Show that ψ is surjective. Find the kernel of ψ.

(b) Show that the stablilizer of each four-fold rotation axis is conjugate to $D_4 \subseteq S_4$.

(c) Show that $L \mapsto \mathrm{Stab}(L)$ is a bijection between the set of four-fold rotation axes and the stabilizer subgroups of these axes in G. This map is G-equivariant, where G acts on the set of stabilizer subgroups by conjugation.

Exercise 5.5.8. Let H be a proper subgroup of a finite group G. Show that G contains an element which is not in any conjugate of H.

Exercise 5.5.9. Find all (2- and 3-) Sylow subgroups of S_4

Exercise 5.5.10. Find all (2- and 3-) Sylow subgroups of A_4

Exercise 5.5.11. Find all (2- and 3-) Sylow subgroups of D_6

Exercise 5.5.12. Let G be a finite group, p a prime and N a normal subgroup of G of order p^s for some s. Show that H is contained in every p-Sylow subgroup of H.

Exercise 5.5.13. Let G be a finite group, p a prime, P a p-Sylow subgroup of G, and N a normal subgroup of G. Show that PN/N is a p-Sylow subgroup of G/N and that $P \cap N$ is a p-Sylow subgroup of N.

Exercise 5.5.14. This is an elaboration of Exercise 5.3.6. Let p and q be primes such that $p > q$ and q divides $p-1$.
One can show that the automorphism group of $\mathbb{Z}_p$ is actually *cyclic* of order $p-1$. We will do this in Section 6.3. Assuming this fact for now, show that there is exactly one non-abelian group of order pq, up to isomorphism, by the following steps:

(a) Show that if G is a non-abelian group of order pq, then G is isomorphic to a semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_q$.

(b) Since $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$, it follows that $\mathrm{Aut}(\mathbb{Z}_p)$ has a *unique* subgroup A of order q.

(c) Suppose α and β are (injective) homomorphisms of $\mathbb{Z}_q$ into $\mathrm{Aut}(\mathbb{Z}_p)$. Then $\alpha(\mathbb{Z}_q) = \beta(\mathbb{Z}_q) = A$.

(d) Write $\varphi = \alpha([1]_q)$ and $\psi = \beta([1]_q)$. Then both φ and ψ are generators of the cyclic group A, so there exist integers a and b such that $\psi = \varphi^a$ and $\varphi = \psi^b$.

(e) Show that $ab \equiv 1 \pmod{q}$, so that $[sab]_q = [s]_q$ for all s.

(f) Show that $\alpha([s]_q) = \beta([sb]_q)$ and $\beta([s]_q) = \alpha([sa]_q)$ for all s.

(g) Consider $\mathbb{Z}_p \underset{\alpha}{\rtimes} \mathbb{Z}_q$ and $\mathbb{Z}_p \underset{\beta}{\rtimes} \mathbb{Z}_q$. Define $\lambda : \mathbb{Z}_p \underset{\alpha}{\rtimes} \mathbb{Z}_q \to \mathbb{Z}_p \underset{\beta}{\rtimes} \mathbb{Z}_q$ by $\lambda(([t]_p, [s]_q)) = ([t]_p, [sb]_q)$, and $\mu : \mathbb{Z}_p \underset{\beta}{\rtimes} \mathbb{Z}_q \to \mathbb{Z}_p \underset{\alpha}{\rtimes} \mathbb{Z}_q$ by $\mu(([t]_p, [s]_q)) = ([t]_p, [sa]_q)$. Show that λ and μ are homomorphisms, and inverses of each other.

It follows that there is, up to isomorphism, exactly one non-abelian group of order pq.

Exercise 5.5.15. Let G be a finite group, p a prime and P a p-Sylow subgroup of G. Show that $N_G(N_G(P)) = N_G(P)$.

Definition 5.5.2. Let p be a prime. A group (not necessarily finite) is called a p-group if every element has finite order p^k for some $k \geq 0$.

Exercise 5.5.16. Show that a finite group G is a p-group if, and only if, the order of G is equal to a power of p.

Exercise 5.5.17. Let N be a normal subgroup of a group G (not necessarily finite). Show that G is a p-group if, and only if, both N and G/N are p-groups.

Exercise 5.5.18. Let H be a subgroup of a finite p-group G, with $H \neq \{e\}$. Show that $H \cap Z(G) \neq \{e\}$.

Exercise 5.5.19. Let G be a finite group, p a prime, and P a p-Sylow subgroup. Suppose H is a *normal* subgroup of G of order p^k for some k. Show that $H \subseteq P$.

Exercise 5.5.20. Show that a group of order $2^n 5^m$, $m, n \geq 1$, has a normal 5-Sylow subgroup. Can you generalize this statement?

Exercise 5.5.21. Show that a group G of order 56 has a normal Sylow subgroup. Hint: Let P be a 7-Sylow subgroup. If P is not normal, count the elements in $\bigcup_{g \in G} gPg^{-1}$.