Exponential and logarithm functions

What are the exponential and logarithm functions and what is the number $e$?
There are a number of approaches to developing the exponential and log functions. The easiest, most elegant and satisfactory approach requires, unfortunately, that one already knows the elements of integral calculus, the second half of this course. But I will outline the approach for you here.

For $t \geq 1$, define $A(t)$ to be the area under the curve $y = 1/x$ between $x = 1$ and $x = t$. The following picture illustrates $A(4)$.

For $t \leq 1$, define $A(t)$ to be the negative of the area under the curve $y = 1/x$ between $x = t$ and $x = 1$. For example, $A(1/3)$ is the negative of the area illustrated in the following picture:

Then one can show, and this is not all that difficult, that the function $A(t)$ has the properties of a logarithm function, namely

\begin{equation}
A(ab) = A(a) + A(b)
\end{equation}

for any positive numbers $a$ and $b$. Moreover, one can show that $\lim_{t \to \infty} A(t) = \infty$ and $\lim_{t \to 0^+} A(t) = -\infty$. 

1
By basic properties of integrals (to be developed later in this course), \( A \) is a differentiable function on its entire domain \((0, \infty)\), and \( A'(t) = 1/t \). Moreover, again by basic properties of integrals, \( A \) is a strictly increasing function whose range is all of the real numbers.

Because \( A \) has the properties of a logarithm function, we call it the natural logarithm and denote it by \( A(t) = \ln(t) \). So, in this approach, \( \ln \) is defined in terms of the area under the curve \( y = 1/x \). We have

\[
(2) \quad \frac{d}{dt} \ln(t) = 1/t.
\]

Because \( \ln \) is a strictly increasing function from \((0, \infty)\) with range \( \mathbb{R} \), it has an inverse function \( \exp : \mathbb{R} \rightarrow (0, \infty) \). Because \( \ln \) has the properties of a logarithm function, it is not at all hard to show that \( \exp \) has the properties of an exponential function, for example

\[
(3) \quad \exp(a + b) = \exp(a) \exp(b),
\]

for all real numbers \( a \) and \( b \).

Because \( \ln \) and \( \exp \) are inverse functions, we have \( \ln(a) = b \) if, and only if, \( \exp(b) = a \). There is a number \( e \) such that \( \ln(e) = 1 \); in fact, \( e \) is the unique number such that the area under the curve \( y = 1/x \) between \( x = 1 \) and \( x = e \) is equal to 1. Since \( \ln(e) = 1 \), we have

\[ \exp(1) = e. \]

Note also that \( \ln(1) = 0 \), so

\[ \exp(0) = 1. \]

For all real numbers \( a \), we have

\[ \exp(a) \exp(-a) = \exp(a + (-a)) = \exp(0) = 1, \]

so

\[
(4) \quad \exp(-a) = \frac{1}{\exp(a)}.
\]

Now using the basic property of the exponential function (equation (3)), we can show that

\[ \exp(m/n) = e^{m/n} \]

for rational numbers \( m/n \), where, on the right hand side, \( e^{m/n} \) is defined in the elementary way, as the \( n \)-th root of the positive number \( e^m \).

At this point we define \( e^x \) as

\[ e^x = \exp(x), \]

for all real numbers \( x \). When \( x \) is rational, this agrees with the elementary meaning of \( e^x \). Finally, it follows from properties of inverse functions, which we will discuss in class, that \( e^x \) is a differentiable function and

\[ \frac{d}{dx} e^x = e^x. \]
All of this may seem rather roundabout, but it is simpler than the supposedly elementary approach in the text, which hides a lot of technical difficulties. Here is the outline of the supposedly elementary approach.

1. Given a positive number $a \neq 1$, one defines $a^{m/n}$ in the elementary way for rational numbers $m/n$. One observes that this function, so far defined only for rational numbers, has the usual properties of an exponential.

2. One shows that $a^x$ is a continuous function of the rational variable $x$. Now one has to show that this function can be extended to a continuous function defined on all real numbers, and that the resulting extension still has the usual properties of an exponential. This involves a lot of technical detail.

3. For convenience, write $f_a(x) = a^x$. One shows that $f_a$ is differentiable and that $f'_a(x) = f'_a(1)f_a(x)$. That is, the derivative of $a^x$ is a constant multiple of $a^x$ and the constant factor is the derivative of $a^x$ at $x = 1$.

4. Now one can show that there is a unique positive number $e$ such that the constant factor in the derivative is 1, so $\frac{d}{dx}e^x = e^x$. This is taken as the definition of $e$.

5. Finally, $\ln$ is defined as the inverse of the function $e^x$. It follows from properties of inverse functions (to be discussed in class) that $\frac{d}{dx}\ln(x) = 1/x$.

There are a lot of difficulties to be overcome in step (2), so this is actually not such an elementary approach.