Assignment 5

$R$ will always denote a unital ring. Modules are left modules, unless otherwise indicated.

(1) Let $X, A, B$ be $R$–modules and $\alpha : X \to A, \beta : X \to B$. Define $C' = (A \oplus B)/N$, where $N = \{ (\alpha(x), -\beta(x)) : x \in X \}$. Define $\alpha' : B \to C'$ by $\alpha'(b) = (0, b) + N$, and $\beta' : A \to C$ by $\beta'(a) = (a, 0) + N$. Show that $\beta' \circ \alpha = \alpha' \circ \beta$. Discover and prove a universal property for the triple $(\alpha', \beta', C')$ relative to the fixed data $(X, A, B, \alpha, \beta)$.

(Hint: Consider $C'', \alpha'' : B \to C'', \beta'' : A \to C''$ such that $\beta'' \circ \alpha = \alpha'' \circ \beta$.) Show that if $\alpha$ is injective, then $\alpha'$ is injective.

(2) Use the construction in question (1) to show that if an $R$–module $Q$ has the property that every short exact sequence $0 \to Q \to B \to C \to 0$ splits, then $Q$ is injective.

(3) Invent and analyze an analogue of problem (1) with all the arrows reversed.

(4) Use your solution to exercise (3) to give a new proof that a module $P$ is projective if it has the property that every short exact sequence $0 \to A \to B \to P \to 0$ splits.

(5) Recall that an $R$–module $N$ is said to be flat if the functor $\_ \otimes_R N$ is exact. Let $M = \bigoplus_j M_j$. Show that $M$ is flat if, and only if, each $M_j$ is flat. (You will use that for every $A$, we have a natural isomorphism $A \otimes M \cong \bigoplus_j (A \otimes M_j)$.) Observe that $R$ is flat as an $R$ module. Conclude that every free $R$ module is flat and that every projective $R$ module is flat.