Do all problems.  
Responses will be judged for accuracy, clarity and coherence.

1. Show that any two bases of a finite dimensional vector space have the same cardinality.

2.  
   (a) Show that a linear transformation on a finite dimensional vector space \( V \) over a field \( K \) determines a finitely generated torsion \( K[x] \)-module structure on \( V \).
   
   (b) Show that two linear transformations are similar if, and only if, they determine isomorphic \( K[x] \)-modules.

3. State, but do not prove, the theorem on the invariant factor decomposition of a finitely generated module over a principal ideal domain. (Note that the module is not assumed to be a torsion module.)

4. Consider the matrix

\[
A = \begin{bmatrix}
2 & -4 & -12 & 17 & 12 \\
0 & -15 & 9 & 68 & 55 \\
0 & 0 & 1 & 0 & 0 \\
0 & -4 & 0 & 18 & 13 \\
0 & 0 & 3 & 0 & 2
\end{bmatrix}
\]

The characteristic polynomial of \( A \) is \( \chi_A(x) = (x - 1)^2(x - 2)^3 \).

(a) Find the Jordan canonical form of \( A \) and find a matrix \( S \) in \( \text{Mat}_5(\mathbb{Q}) \) such that \( S^{-1}AS \) is in Jordan canonical form.

   It is helpful to know that a basis of the solution space of \((A - E)v = 0\) is

\[
\begin{bmatrix}
-36 \\
39 \\
-4 \\
0 \\
12
\end{bmatrix}
\]

and a basis of the solution space of \((A - 2E)v = 0\) is

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(b) What are the elementary divisors and the invariant factors of \( A \)?

(c) What is the minimal polynomial of \( A \)?

(d) What is the rational canonical form of \( A \)?